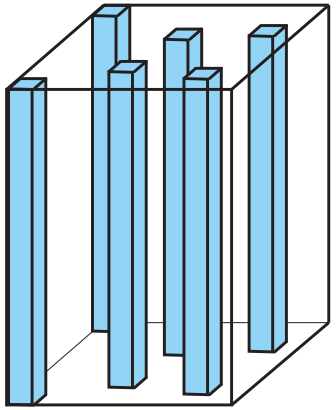
Practical Leverage Score-Based Sampling for Low-Rank Tensor Decompositions

Brett Larsen
Stanford University
DOE Computational Science Graduate Fellow

Joint work with:
Tamara G. Kolda
Sandia National Laboratories

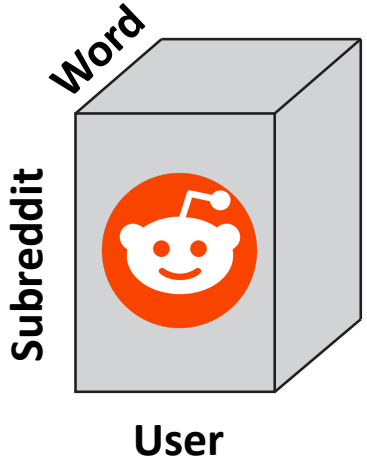


Tamara G. Kolda

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA-0003525.

Example Sparse Multiway Data: Reddit

- Reddit is an American social news aggregator, web content rating, and discussion website
  - A "subreddit" is a discussion forum on a particular topic
- Tensor obtained from frost.io (http://frostt.io/tensors/reddit-2015/)
  - Build from reddit comments posted in the year 2015
  - Users and words with less than 5 entries have been removed



Reddit Tensor

8 million users
200 thousand subreddits
8 million words

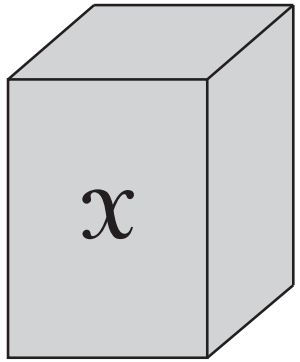
4.7 billion non-zeros ($10^{-8}\%$)
106 gigabytes

$x(i,j,k) = \log$ (1 + the number of times user i used word j in subreddit k)

Used a rank $\boldsymbol{r} = \mathbf{25}$ decompsition

Smith et al (2017). "FROSTT: The Formidable Open Repository of Sparse Tensors and Tools"

CP Decomposition into Rank-1 Components

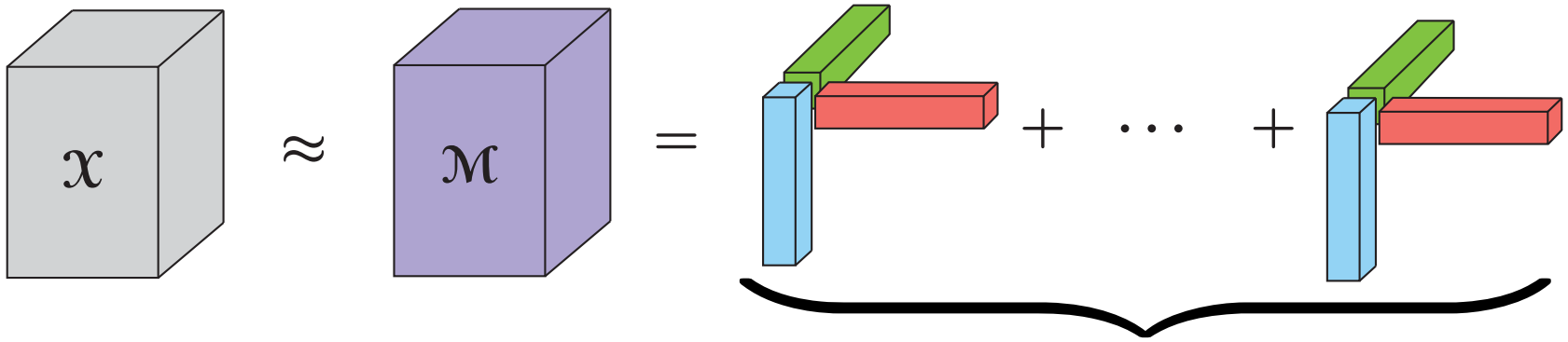


Tensor Properties

Order: $3 \mid \mathcal{X}$ is a 3-way tensor.

Length of each dimension: n_1, n_2, n_3

$$\mathcal{X} \text{ is } n_1 \times n_2 \times n_3$$



Rank r Decomposition

Tensor Decomposition Identifies Factors

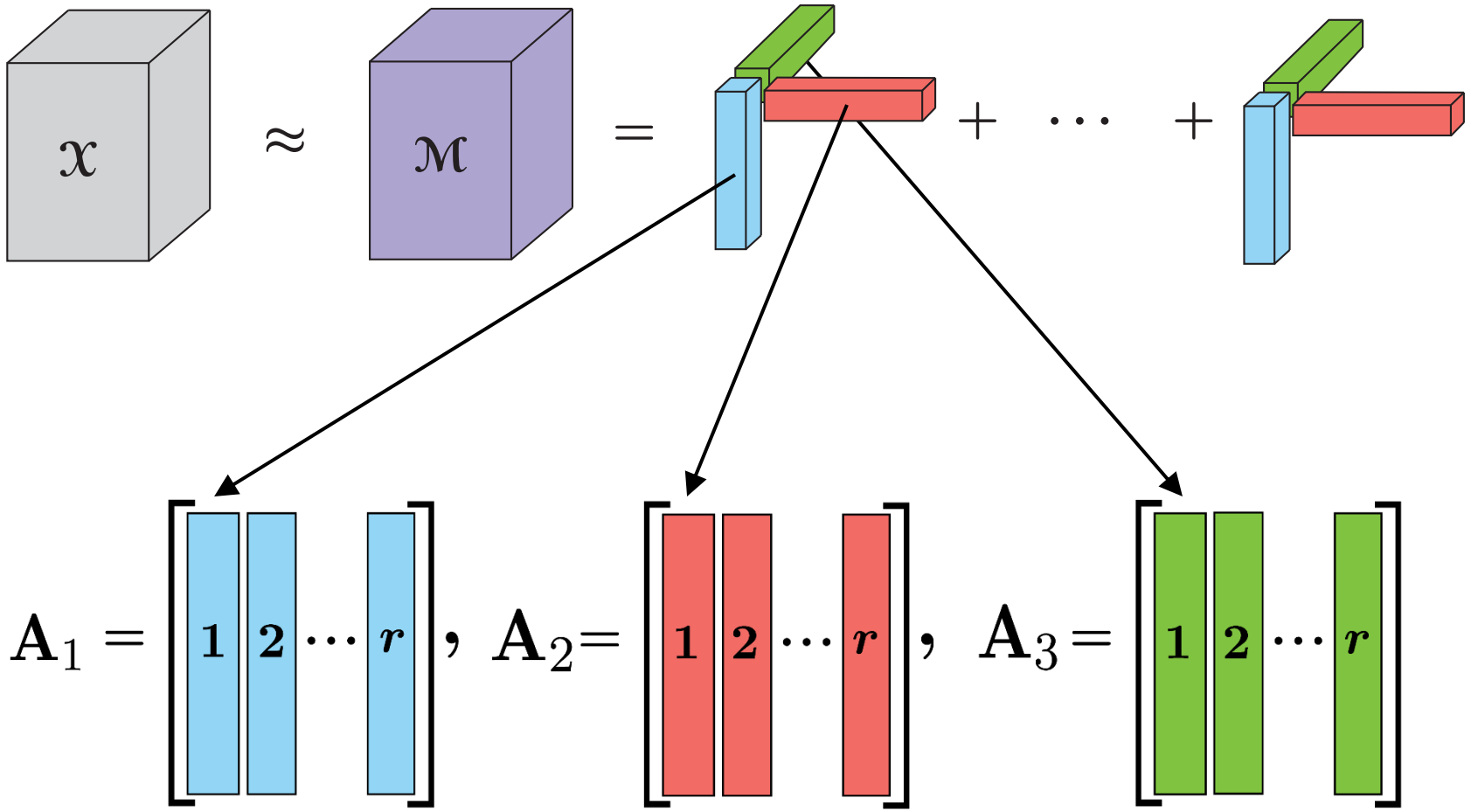
Multi-Index Notation

$$x_i \equiv \mathcal{X}(i_1, i_2, i_3)$$

Ω: Set of all multi-indices

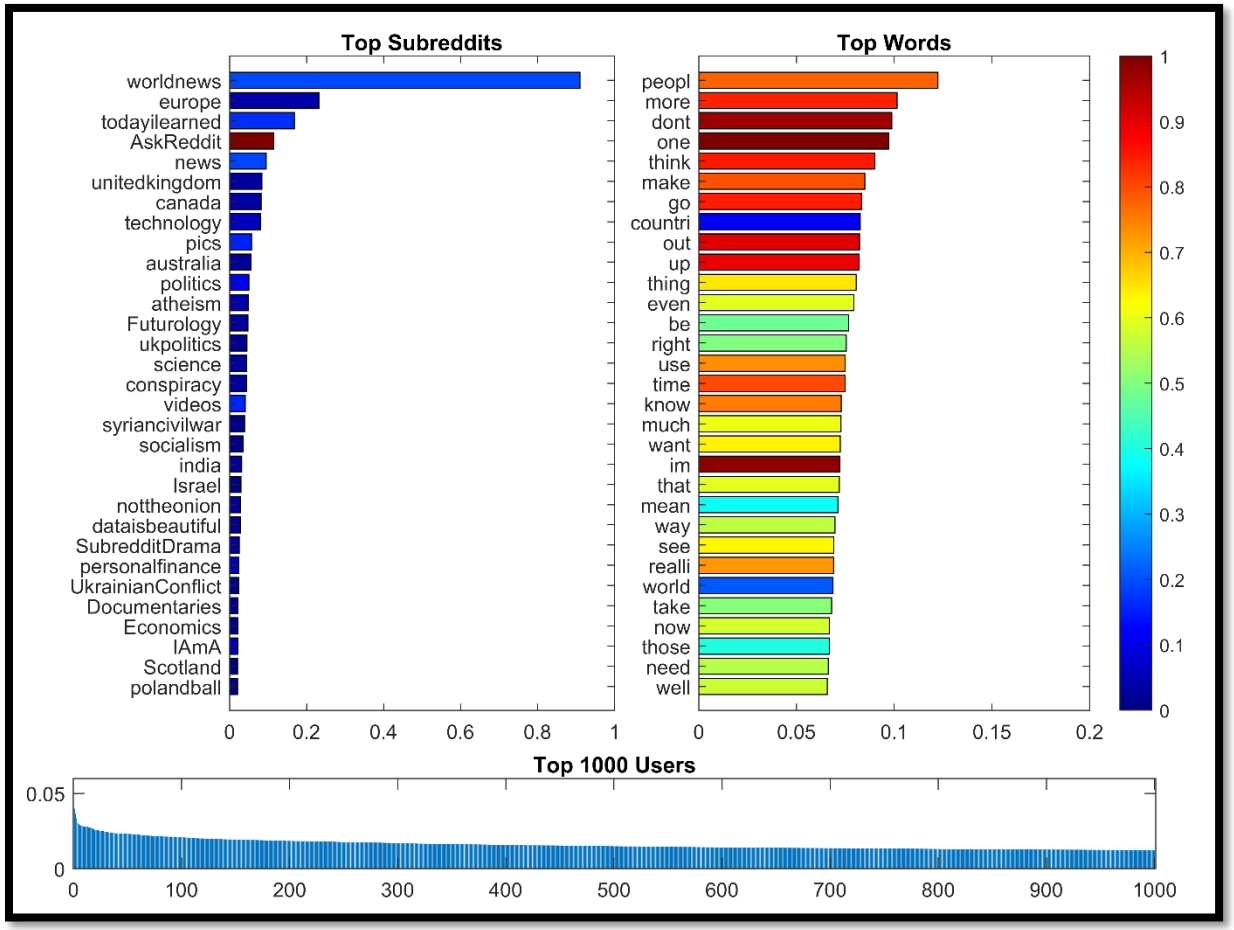
CP Decomposition

$$x_i \approx \sum_{j=1}^{r} \mathbf{A}_1(i_1, j) \ldots \mathbf{A}_3(i_3, j)$$

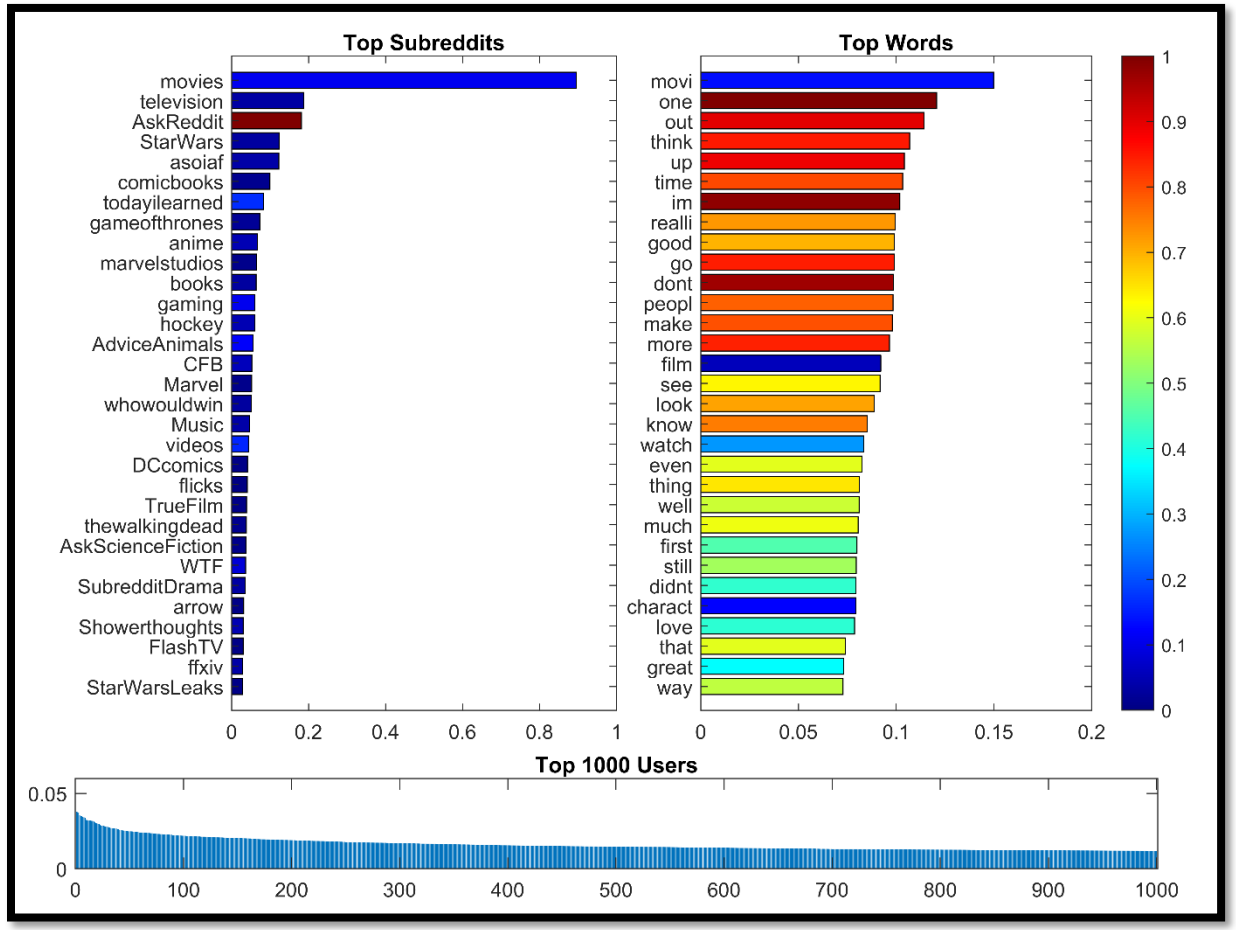


Example Reddit Components Include Rare Words Apropos to High-Scoring Subreddits

Component #6: International News

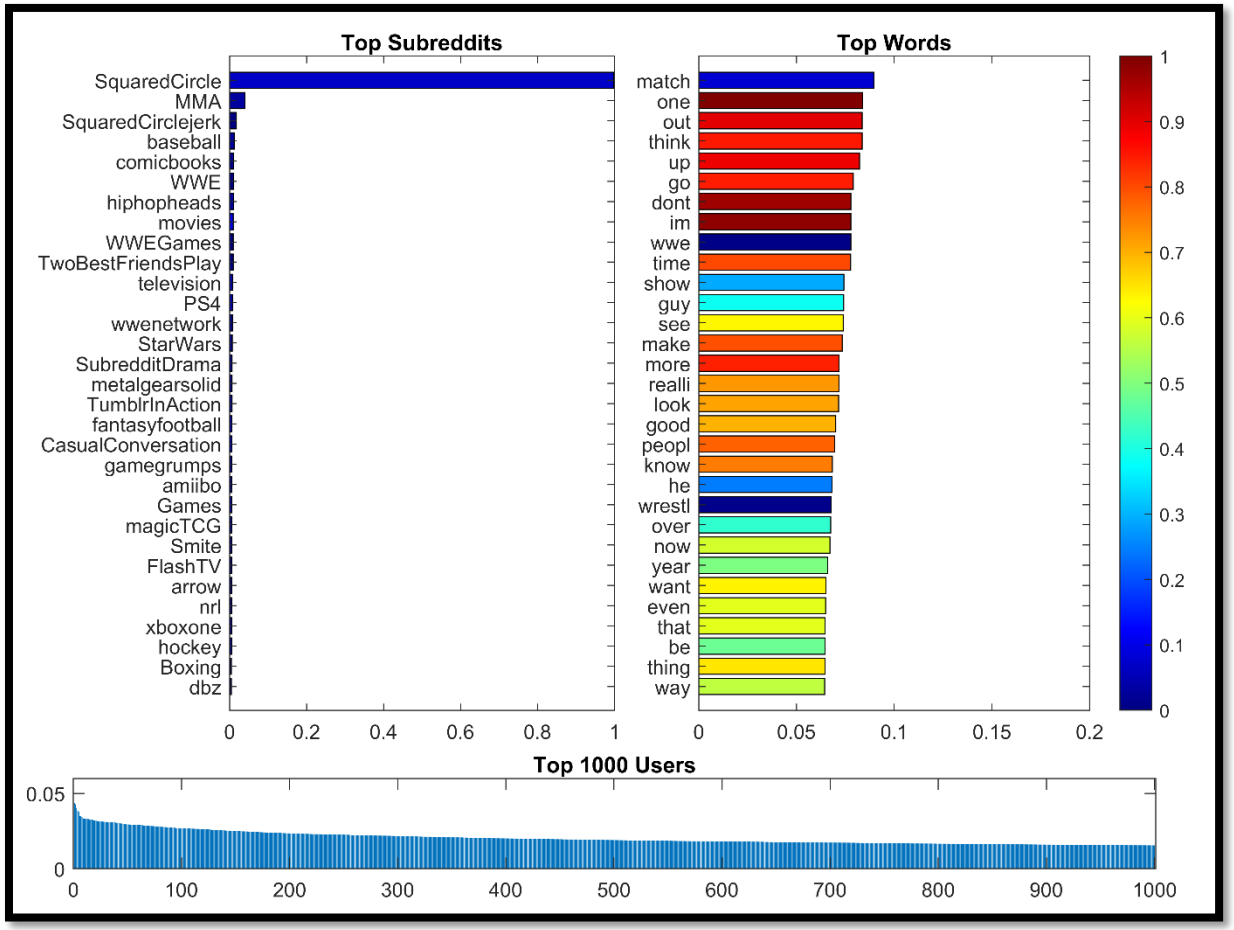


Component #19: Movies & TV

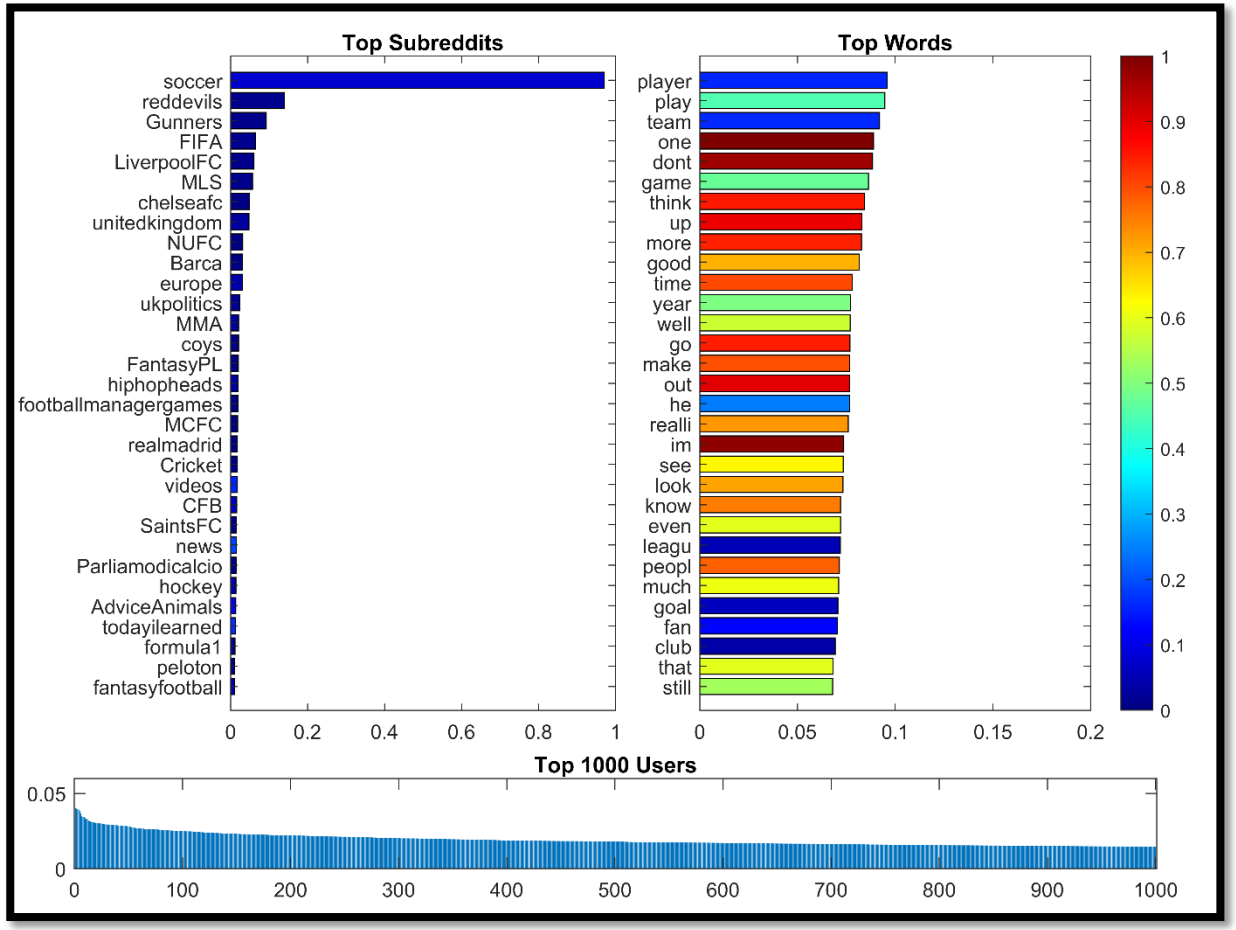


Example Reddit Components Include Rare Words Apropos to High-Scoring Subreddits

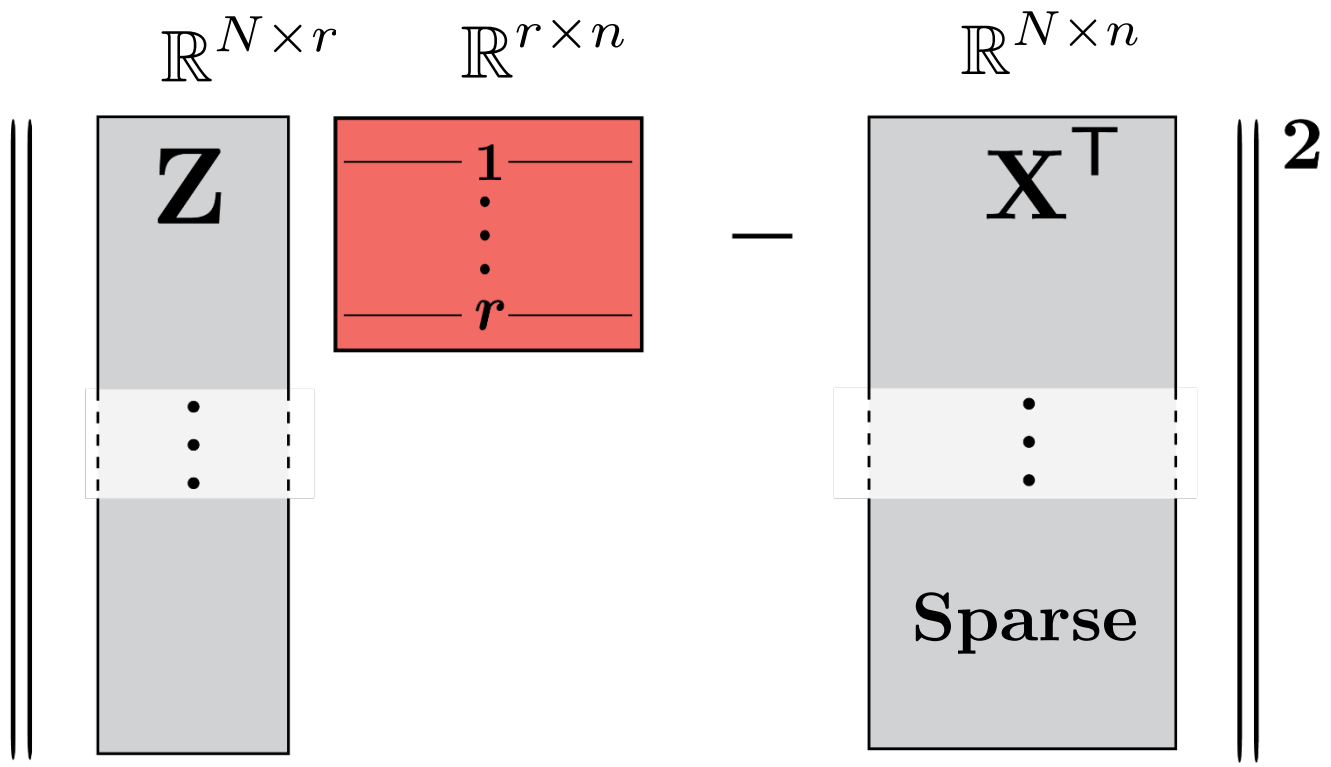
Component #15: Wrestling



Component #18: Soccer



Prototypical CP Least Squares Problem has Khatri-Rao Product (KRP) Structure



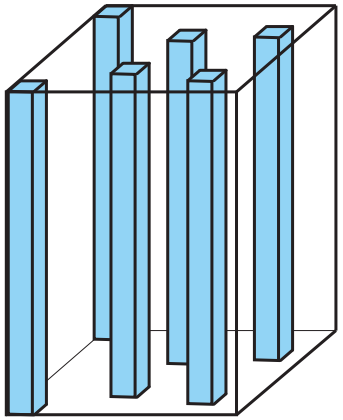
$$\min_{\mathbf{A}} \|\mathbf{Z}\,\mathbf{A}^\mathsf{T} - \mathbf{X}^\mathsf{T}\|^2$$

Alternate between factor matrices.

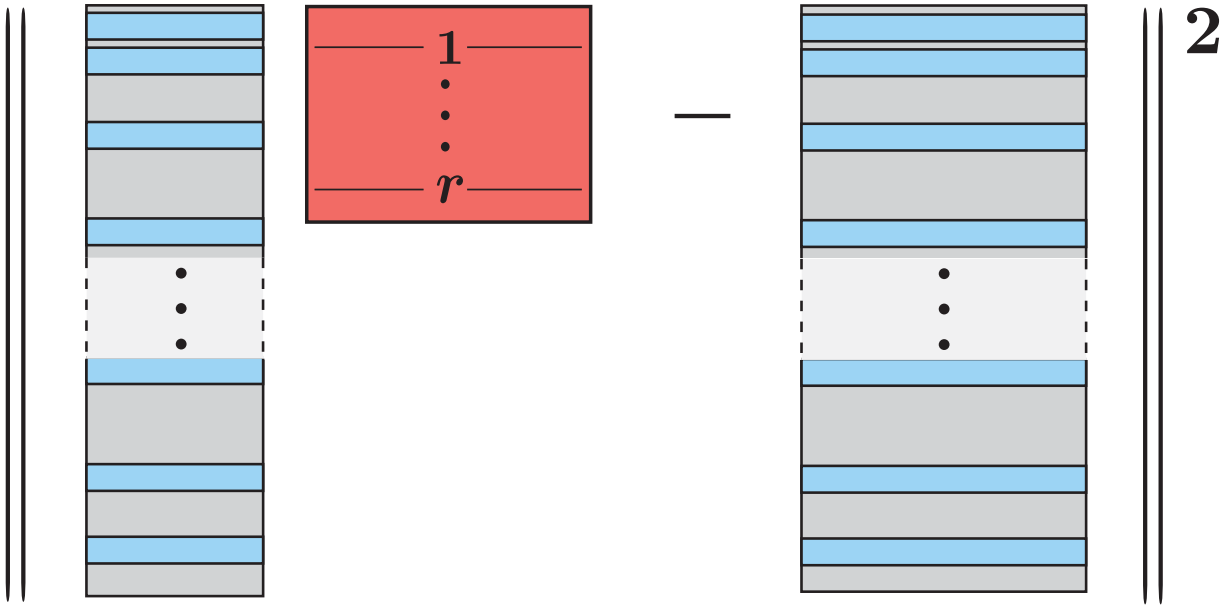
- Subproblem for CP-ALS
  - Solve $d+1$ such subproblems per outer iteration
  - Size n = one of the tensor dimensions
- Matrix $\mathbf{Z}$ (from the "other" factor matrices)
  - Khatri-Rao product (KRP) of d factor matrices
  - $N = O(n^d)$, i.e. very large
  - Structure of this matrix is key!
- Matrix $\mathbf{X}^\mathsf{T}$ (from the data tensor)
  - Transpose of mode-unfolding of tensor
  - Sparse if tensor is sparse, including all-zero rows
  - Cannot afford any transform that destroys sparsity
- Matrix $\mathbf{A}^\mathsf{T}$ (transpose of factor matrix)

CP-ARLS Leverage Applies Sketching to CP Least Squares Problem

How can we utilize sketching in the inner least squares problem of CP-ALS?



Corresponds to sampling fibers from the tensor



$$\min_{\mathbf{A}} \left\| \mathbf{S}\mathbf{Z} \quad \mathbf{A}^{\mathsf{T}} \quad - \quad \mathbf{S}\mathbf{X}^{\mathsf{T}} \right\|^2$$

Battaglino, Balladrd, and Kolda (2017). "A Practical Randomized CP Tensor Decomposition"

Ingredient #1: Leverage Scores Key to Limiting Samples (But too Expensive to Compute)

Leverage score: $\mathbf{Z} \in \mathbb{R}^{N \times r}$

"Economy" SVD:

$$\mathbf{Z} = \boxed{\mathbf{U}}\mathbf{\Sigma}\mathbf{V}^T$$

Spans column space of $\mathbf{Z}$

Leverage score of row i:

$$\ell_i(\mathbf{Z}) = \|\mathbf{U}(i,:)\|_2^2 \in [0,1]$$

Leverage scores indicate which rows most influence solution but cost $O(Nr^2)$ to compute

Rough Intuition:

$$\ell_1(\mathbf{Z}) = 1$$

$$\sum \ell_i(\mathbf{Z}) = r$$

How "important" a row is to column space.

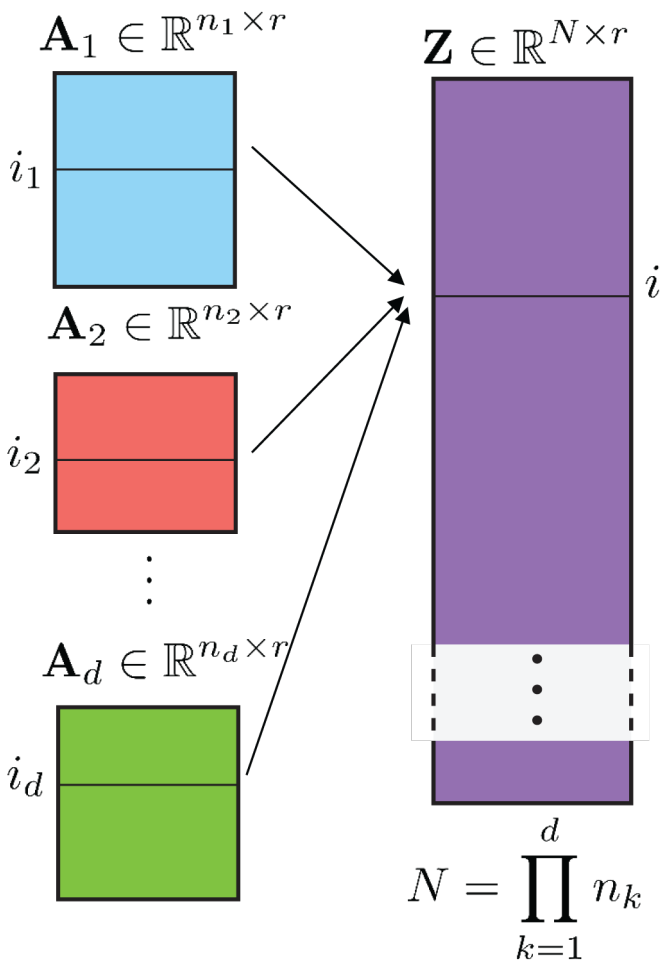
Can use approximate leverage score, pay penalty in required sample number

$$\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1r} \\ 0 & z_{22} & \cdots & z_{2r} \\ 0 & z_{32} & \cdots & z_{3r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & z_{N2} & \cdots & z_{Nr} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \vdots \\ \nu_N \end{bmatrix}$$

Survey: D. P. Woodruff, ***Sketching as a Tool for Numerical Linear Algebra***, 2014

Ingredient #2: Exploit KRP Structure to Bound Leverage Scores

$$\mathbf{Z} = \mathbf{A}_d \odot \cdots \odot \mathbf{A}_1$$



Upper Bound on Leverage Score

Lemma (Cheng et al., NIPS 2016; Battaglino et al., SIMAX 2018):

$$\ell_i(\mathbf{Z}) \leq \prod_{k=1}^{d} \ell_{i_k}(\mathbf{A}_k)$$

Too expensive to calculate $O(Nr^2)$

Cheap to calculate individual leverage scores $O(r^2 \sum_k n_k)$

Set probability of sampling row i to:

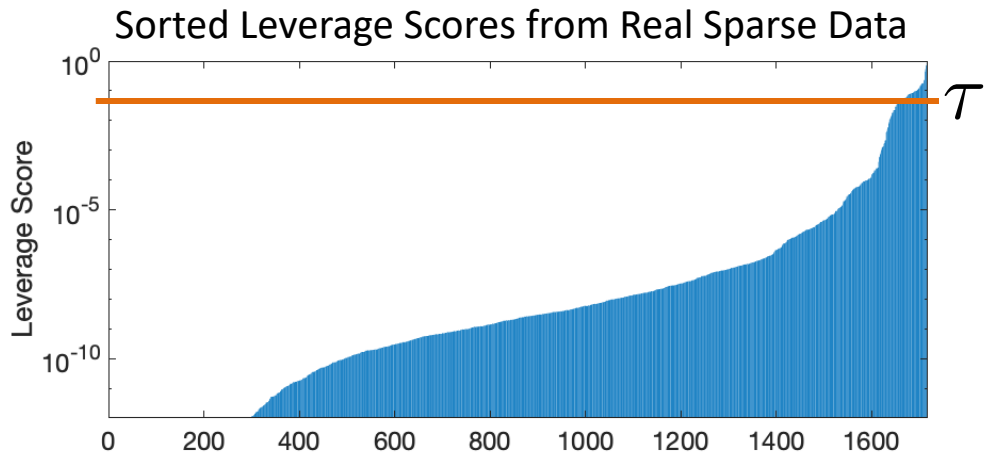
$$p_i \equiv \frac{1}{r^d} \prod_{k=1}^{d} \ell_{i_k}(\mathbf{A}_k)$$

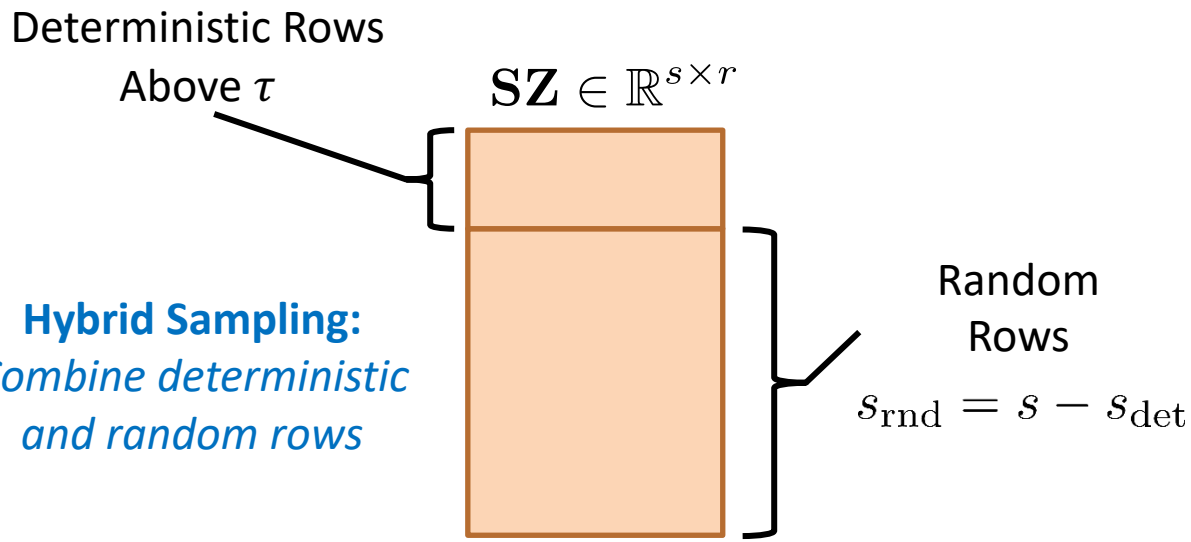
Sample row from each factor matrix to avoid forming all N possible combinations corresponding to rows of $\mathbf{Z}$!

1-1 Correspondence between *linear index* and *multi index:*

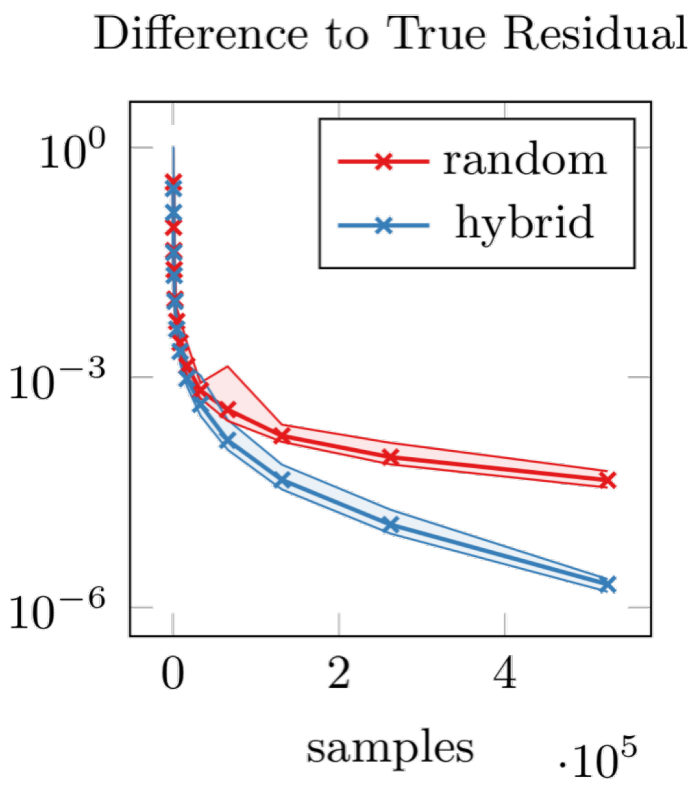
$$i \in [N] \Leftrightarrow (i_1, \ldots, i_d) \in [n_1] \otimes \cdots \otimes [n_d]$$

Ingredient #3: Deterministically Include All High-Probability Rows



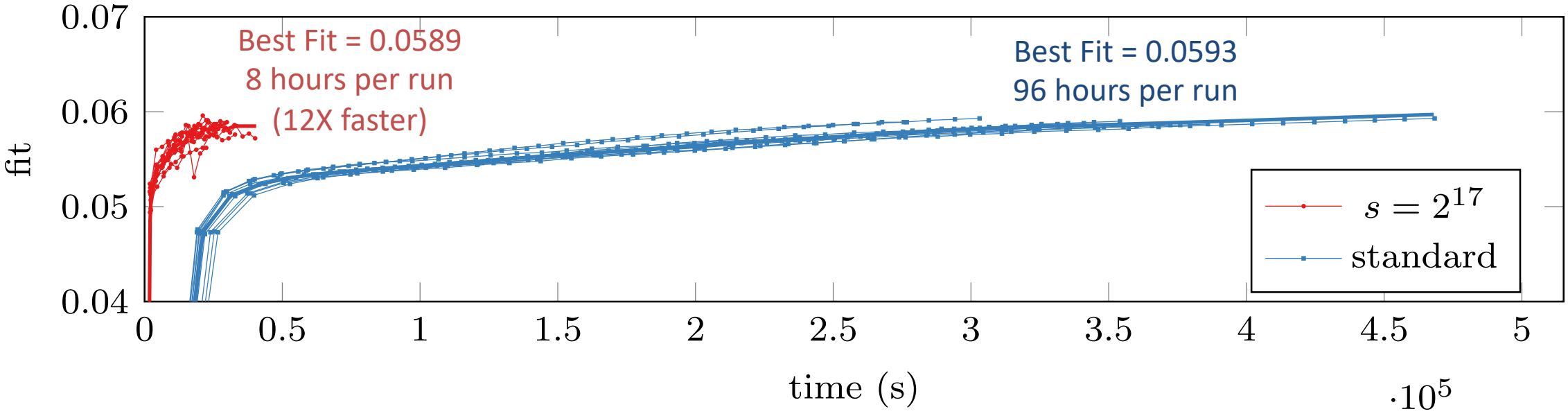


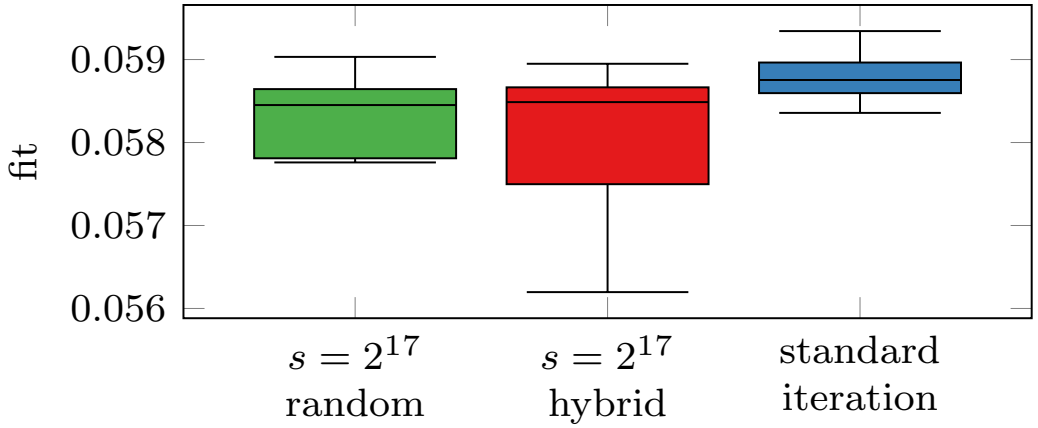
Hybrid Sampling:
Combine deterministic and random rows



Single Least Squares Problem with
N = 46M rows, r = 10 columns,
n = 183 right-hand sides

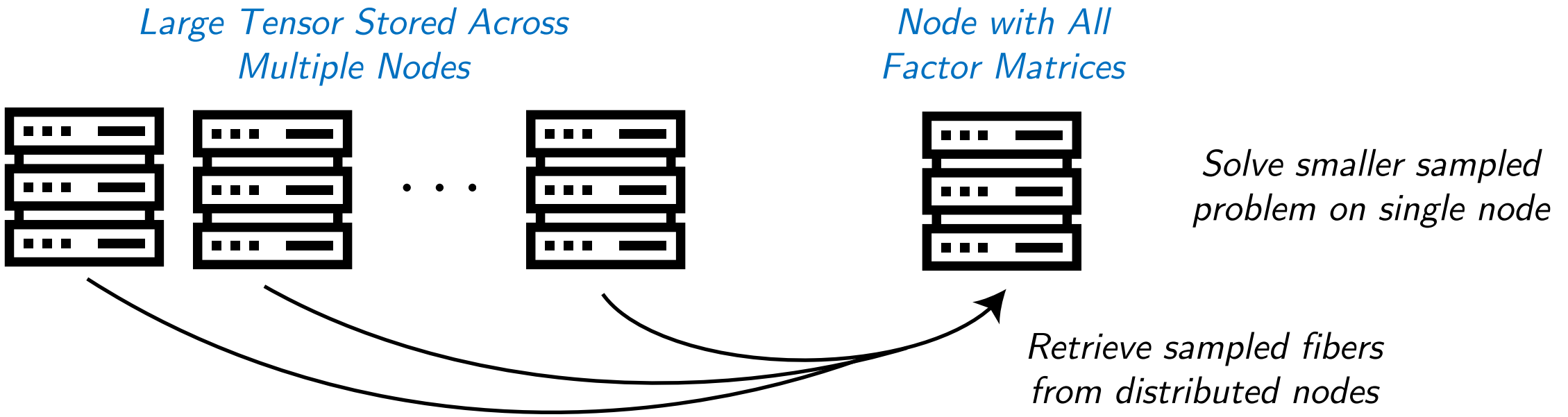
Numerical Results on Reddit Tensor





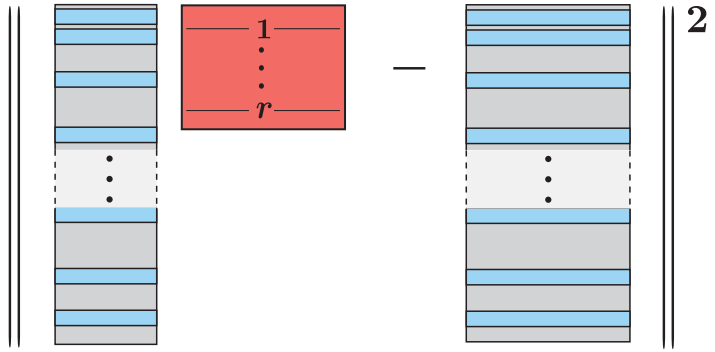
# Samples (S)	Speedup
2^{17} Random	16.08
2^{17} Hybrid	11.87
CP-ALS	1.00

Extensions to Distributed Data and Summary



Summary: Sketching for Sparse Tensor Problems

1. Sampling by leverage scores minimizes required samples
2. KRP structure enables efficient leverage score estimates
3. Hybrid sampling for concentrated leverage scores



Questions

B.W. Larsen and T.G. Kolda. *"Practical Leverage-Based Sampling for Low-Rank Tensor Decomposition"* on arXiv.