A Nonhydrostatic Isopycnal-Coordinate Ocean Model

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Abstract:

Numerical models of small-scale ocean processes such as internal waves require computationally expensive **nonhydrostatic models** and high horizontal resolution.

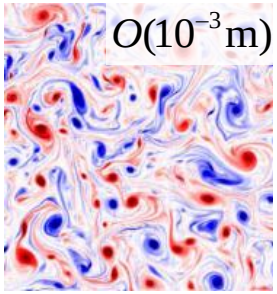
The horizontal grid spacing (Δx) must be less than the internal interface depth (h_1).

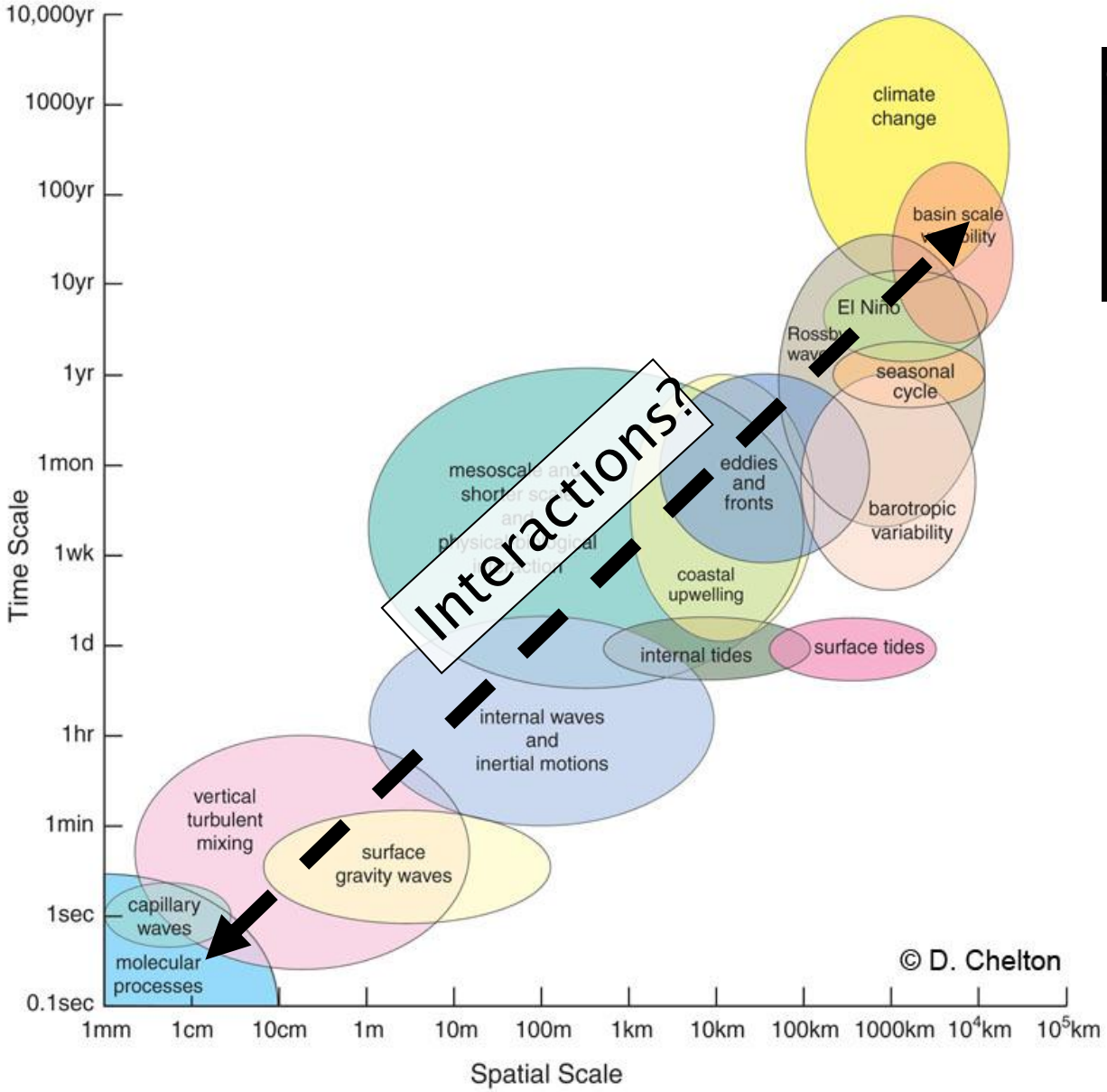
Isopycnal coordinates provide a natural representation of a stratified fluid which in turn reduces the number of vertical grid points and thus improves efficiency.

Existing isopycnal-coordinate models use the **hydrostatic approximation**.

Here, I demonstrate a new nonhydrostatic isopycnal-coordinate ocean model capable of simulating variety of nonhydrostatic ocean processes.

The Ocean: *The* Multiscale Problem

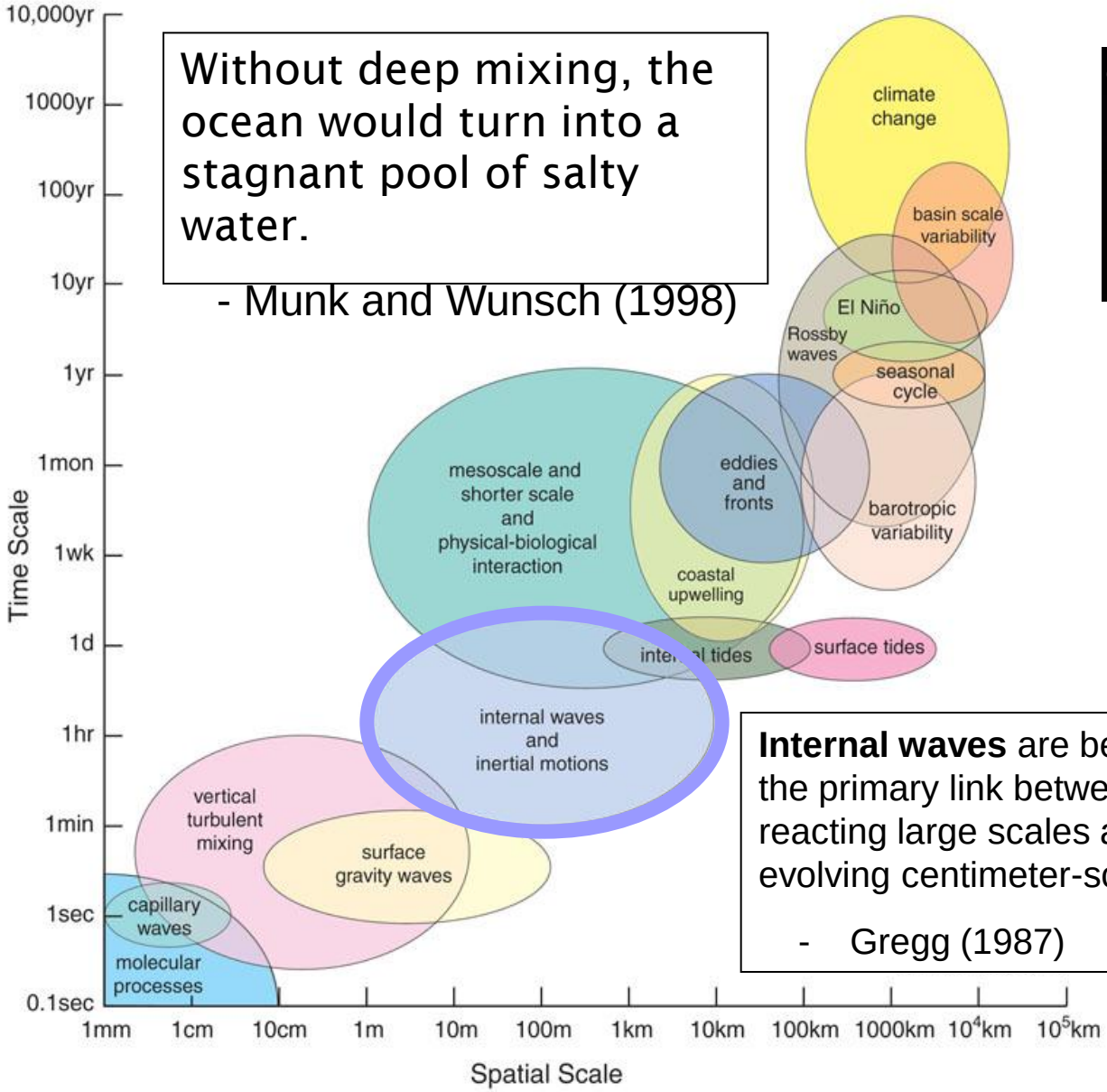


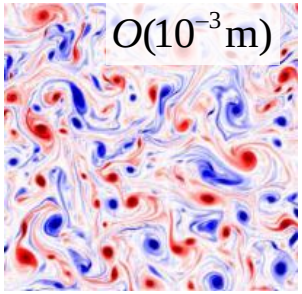


The Ocean: *The* Multiscale Problem

Without deep mixing, the ocean would turn into a stagnant pool of salty water.

- Munk and Wunsch (1998)



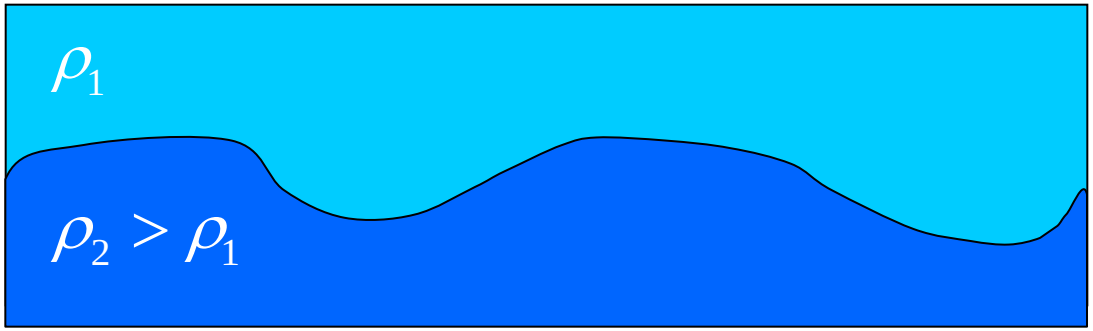


Internal waves are believed to be the primary link between the slowly reacting large scales and the rapidly evolving centimeter-scale mixing.

- Gregg (1987)

Background:

Internal waves are disturbances that propagate in the interior of a stratified fluid (usually on an interface of density variation or "pycnocline").



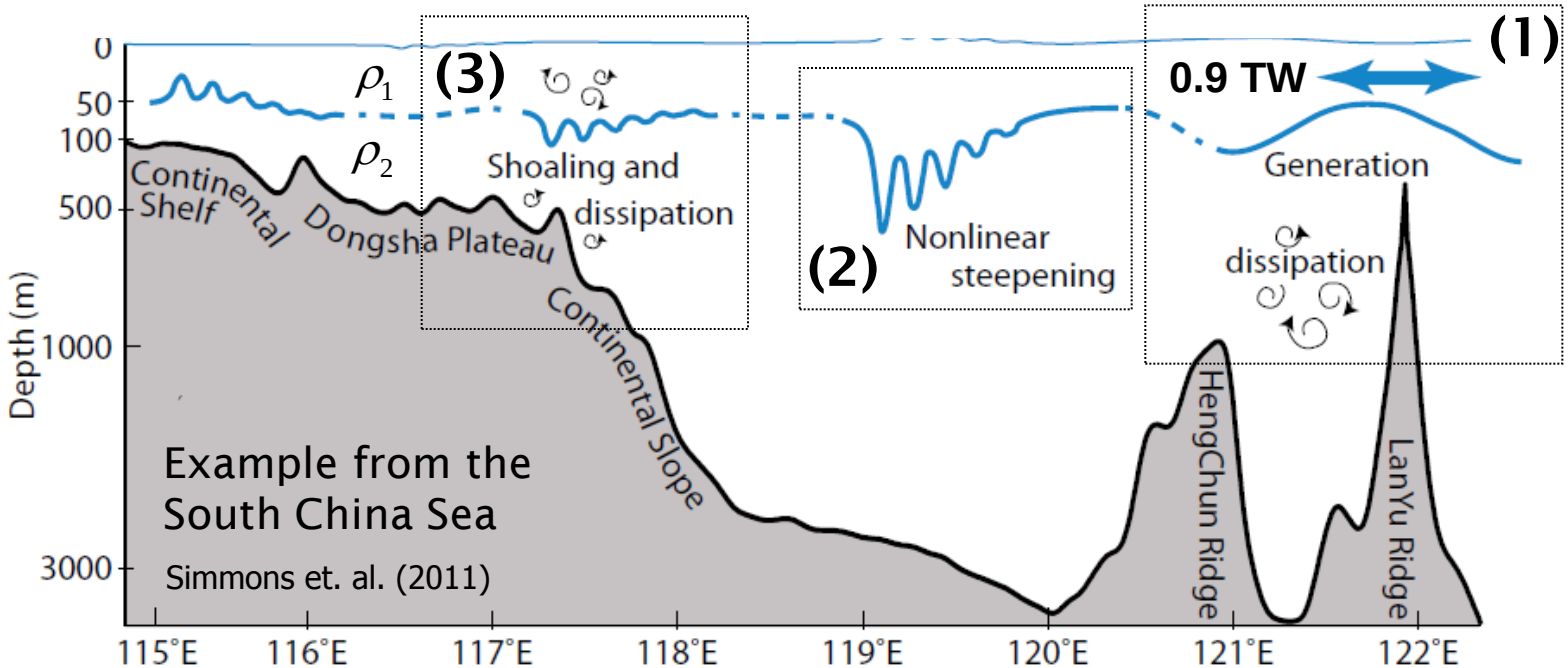
- Important to physical and ecological processes
- Very difficult to model

  - range of scales: O(1 m) turbulence
    O(1 km) wavelengths
    O(1000 km) propagation distances

  - **nonhydrostatic** (computationally expensive – requires 3-D matrix inverse)

Internal Waves

Life-cycle of internal waves:

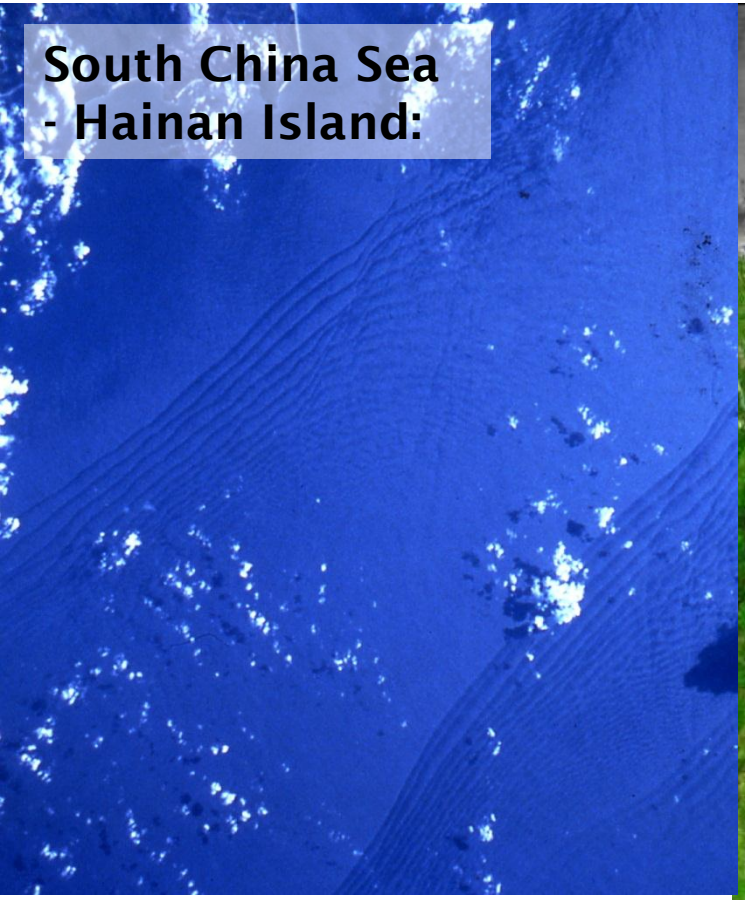


(1) Generation: Tidal flow over ridges and bathymetric features

~ 0.9 TW of tidal power is converted into internal waves

(2) Propagation: Steepening and formation of solitary-like waves Nonhydrostatic effects are important: balance of nonlinearity and dispersion (nonhydrostasy)

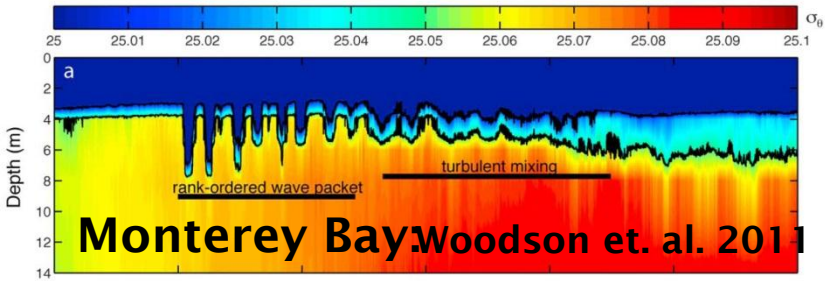
Internal solitary-like waves - “Solitons” found in nature:



Internal Wave Atlas 2004

Alpers et. al. 2010

Li et. al. 2011



Monterey Bay: Woodson et. al. 2011

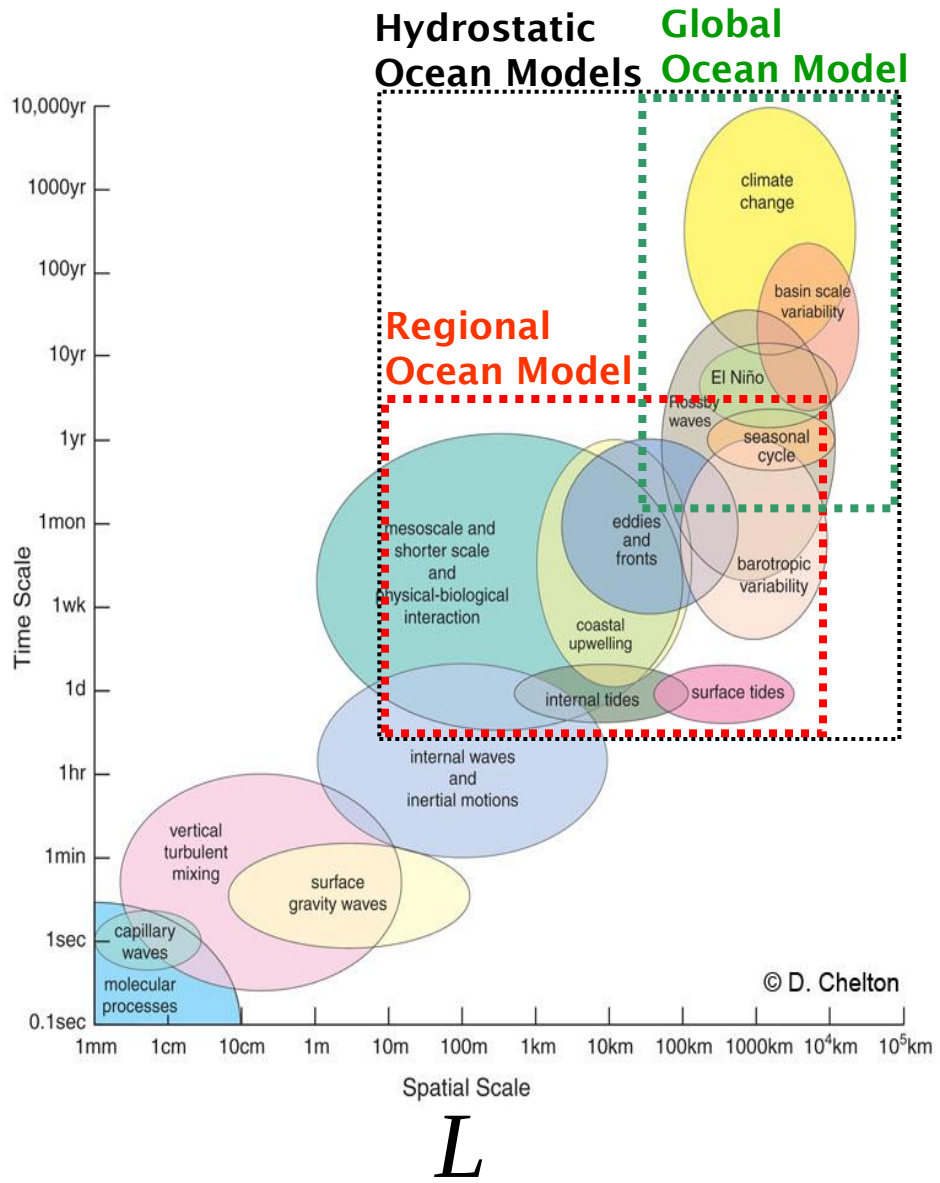
Numerical models & their approximations

Hydrostatic Ocean Models

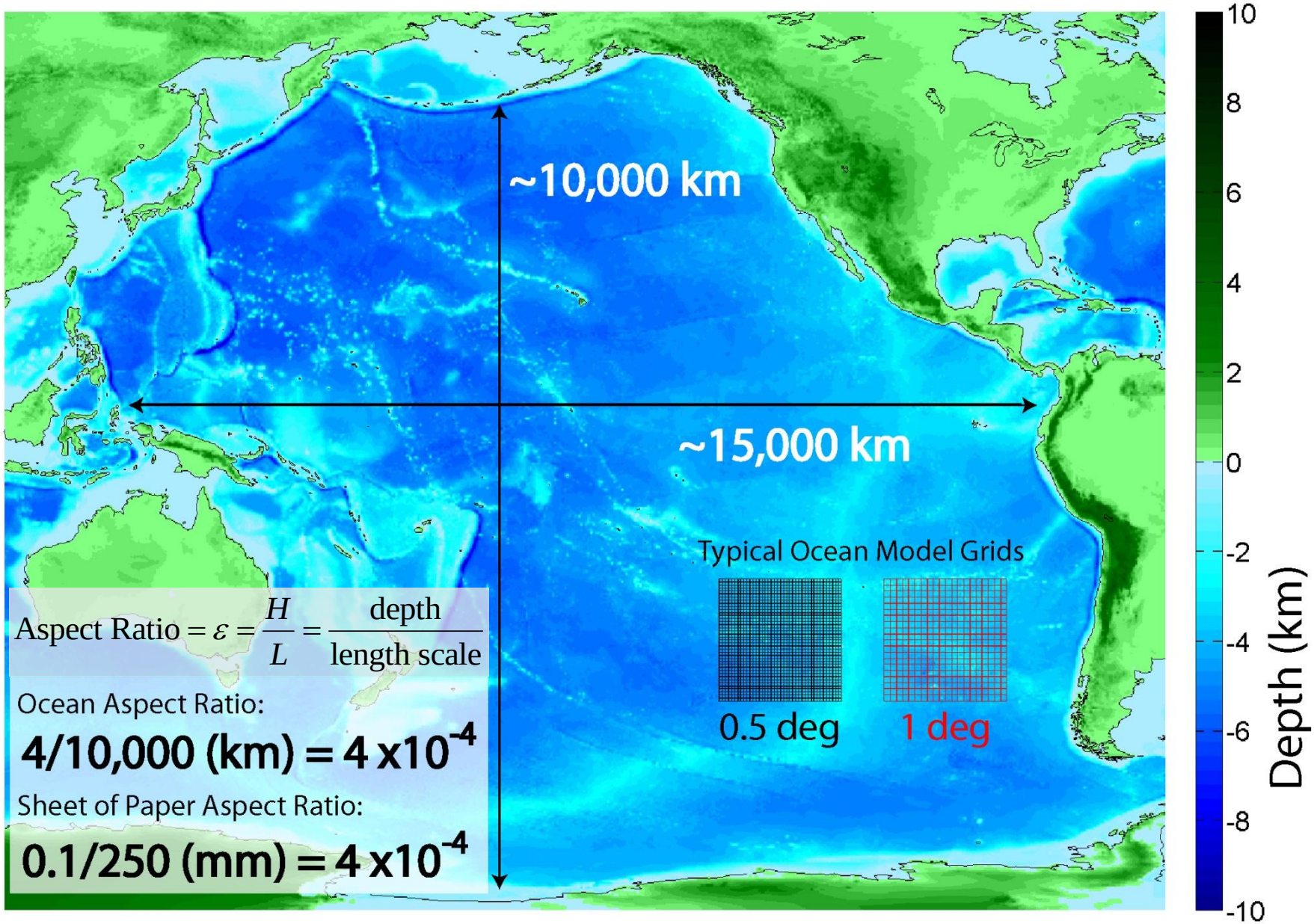
- Hydrostatic approx. is a cornerstone of ocean modeling
- Valid for low aspect-ratio flows

$$H \square L$$

H = depth
L = horizontal length scale



Scaling arguments



Numerical models & their approximations

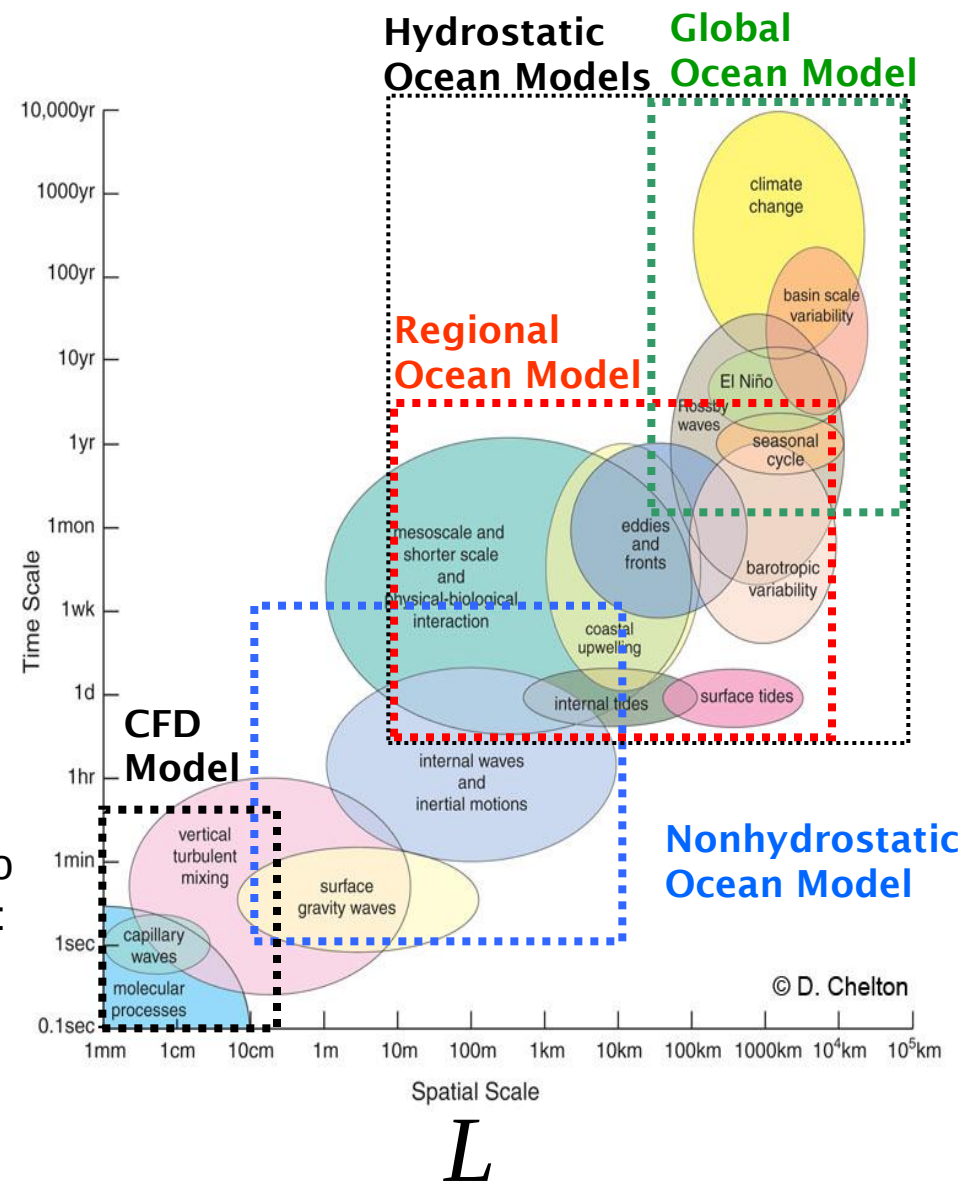
Hydrostatic Ocean Models

- Hydrostatic approx. is a cornerstone of ocean modeling
- Valid for low aspect-ratio flows $H \ll L$

H = depth
L = horizontal length scale

Nonhydrostatic (NH) Ocean Model

- Valid for any aspect-ratio $H \sim L$
- Solve incompressible Navier-Stokes eqs. with a free surface.
- First NH ocean model developed in 1996 to overcome limitations of hydrostatic models:
  - hydrostatic primitive equations are ill-posed with open BC's
  - hydrostatic models are inaccurate for modeling small-scale processes

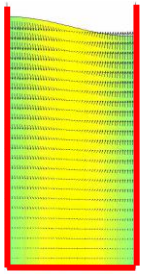


Hydrostatic vs. Nonhydrostatic mode

- Simplified vertical momentum eq.

$$\frac{\partial p}{\partial z} = -\rho g$$

- Waves are **non-dispersive** $c = \sqrt{gH}$
- Limited or no vertical variability



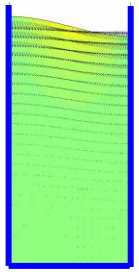
u independent of z

- **Computationally efficient!**
  - Equations are hyperbolic
  - No elliptic solve required (i.e. no linear algebra)

- Full vertical momentum eq.

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0} g + \nu \nabla^2 w$$

- Waves are **dispersive** $c = \sqrt{\frac{g}{k}\tanh(kH)}$
- Proper treatment of vertical variability



$u(z)$

- **Computationally expensive!**
  - requires solution of an elliptic equation
  - requires high horizontal resolution:

$$\boxed{\Delta x < h_1}$$

Δx = horizontal grid spacing

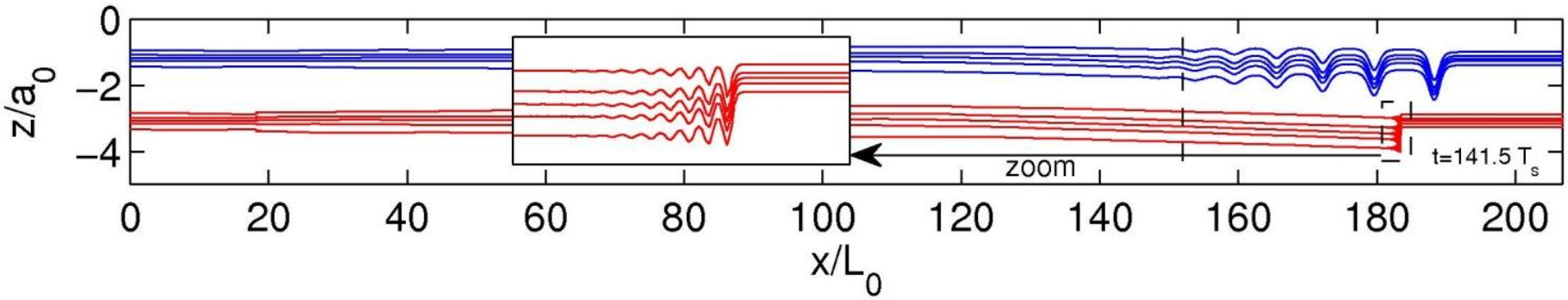
h_1 = internal interface depth O(10-100 m)

to ensure numerical dispersion < physical dispersion for 2nd order accurate ocean models (…most of them)

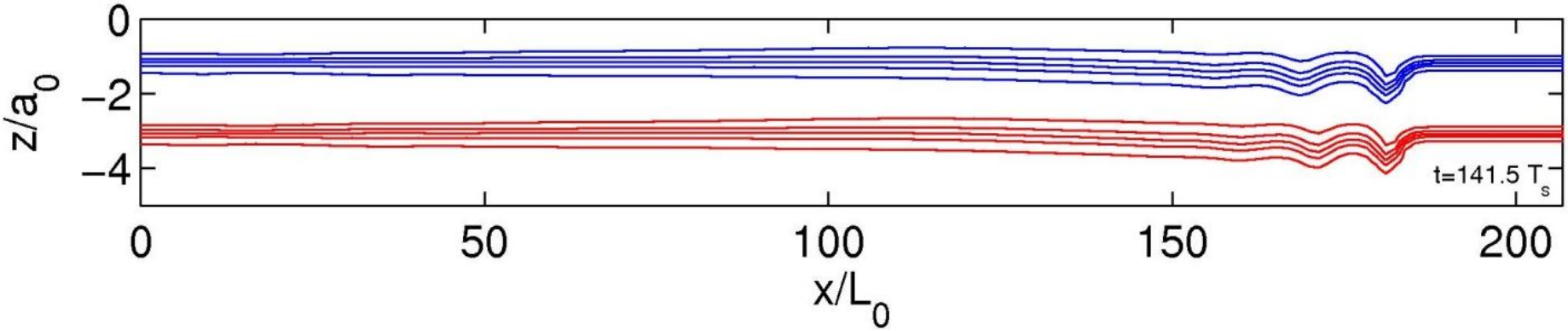
Hydrostatic vs. Nonhydrostatic (SUNTANS)

High Resolution: $\Delta x = h_1 / 4$

Vitousek & Fringer (2011)



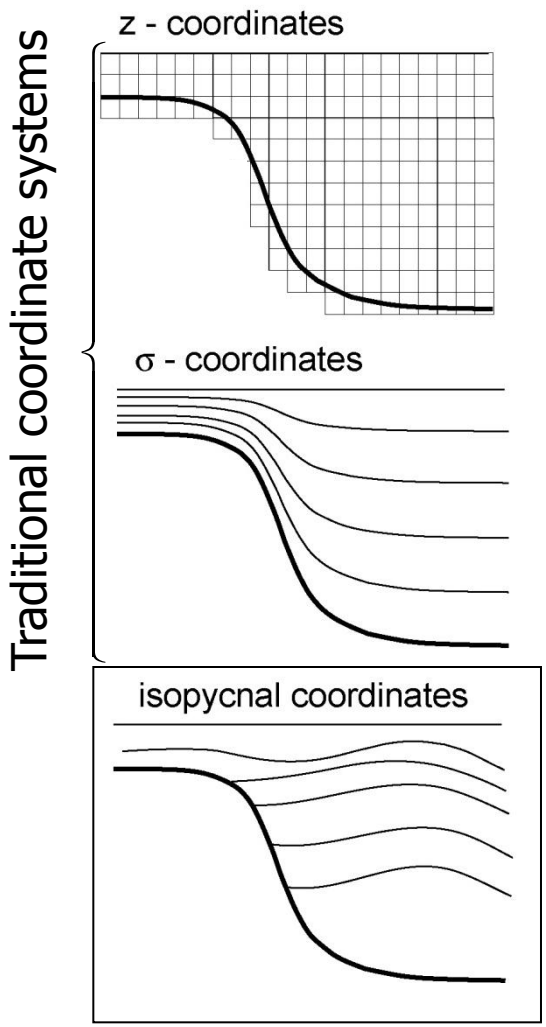
Low Resolution: $\Delta x = 8h_1$



$$\boxed{\Delta x < h_1}$$

It becomes necessary to find other means of reducing resolution requirements and computational cost.

Isopycnal = constant density



Advantages:

- Natural representation of a stratified fluid
  - reduces the number of vertical grid points from O(100) in traditional coordinates to O(1-10)
- No spurious vertical (diapycnal) diffusion/mixing

Challenges:

- Cannot represent unstable stratification.
- Layer outcropping (drying of layers) requires special numerical schemes
- Hydrostatic

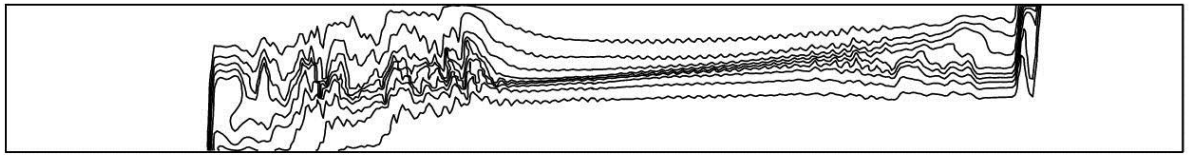
What about a nonhydrostatic isopycnal ocean model?

"**Common sense** suggests that pure isopycnal coordinates should **not** be used for **non-hydrostatic modeling** …

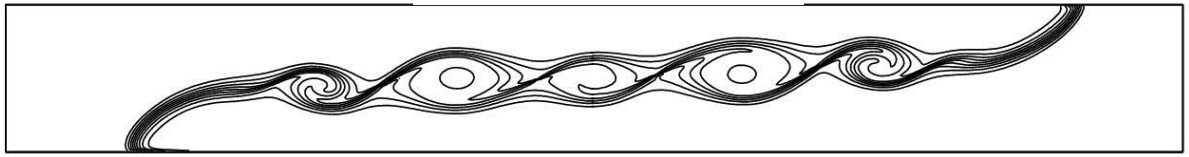
This is, of course, **pointless** since **non-hydrostatic** effects tend to be associated with over turning (e.g. Kelvin–Helmholtz instabilities)."

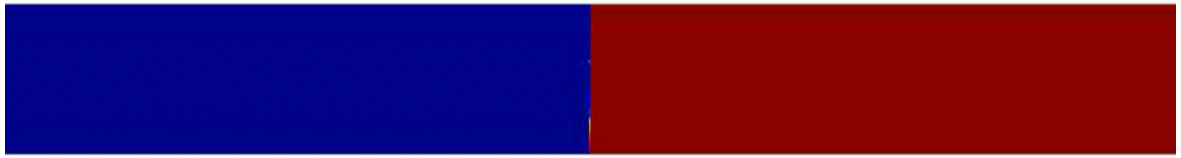
- Adcroft & Hallberg (2006)

Hydrostatic

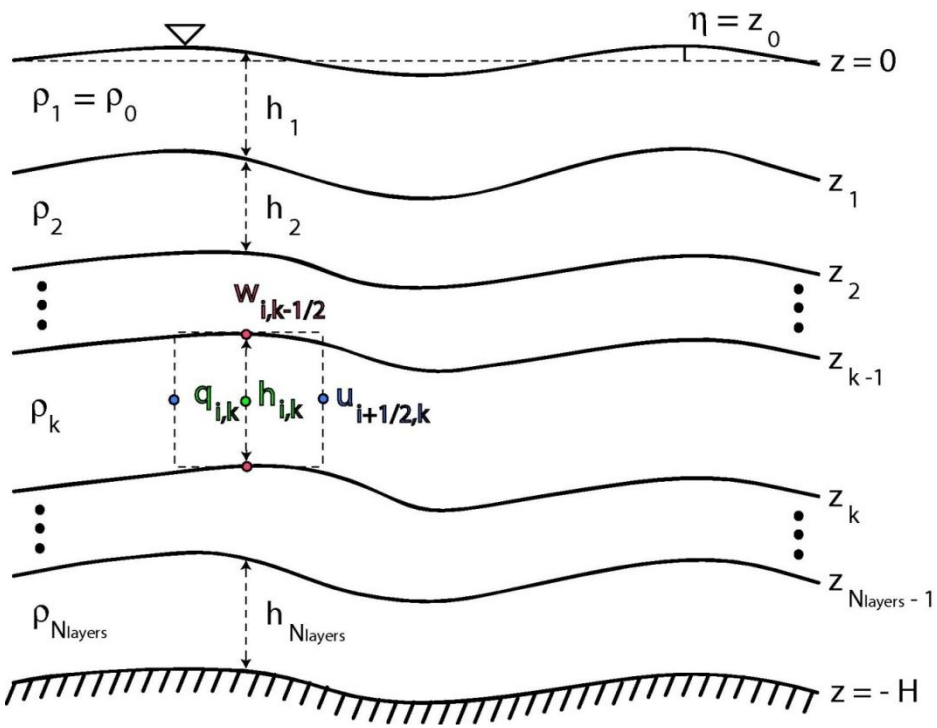


Nonhydrostatic





A Nonhydrostatic Isopycnal Coordinate Ocean Model



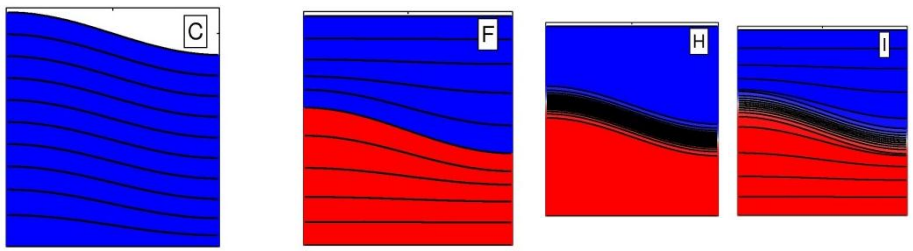
Notes:

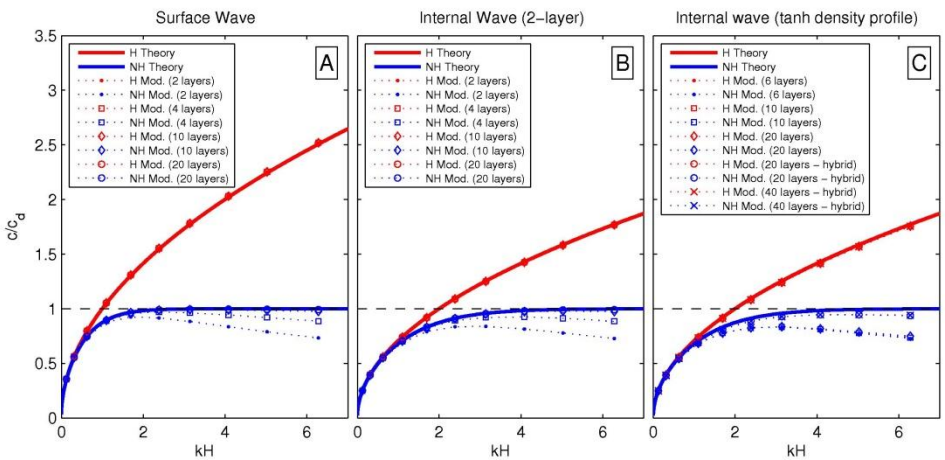
- Solves incompressible Navier-Stokes eqs with a free surface in isopycnal coordinate
- Second-order accurate in time and space
- Staggered C-grid layout
- Split Montgomery potential into Barotropic (implicit) & Baroclinic (explicit) parts
- IMplicit-EXplicit (IMEX) multistep method. Durran (2012)
- θ - method for (implicit) vertical diffusio
- Explicit horizontal diffusion
- Pressure projection (1st order) or Pressure correction (2nd order accuracy)

Test Cases

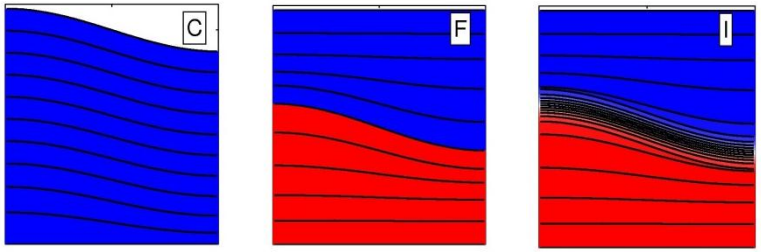
Model comparison with analytical solutions:

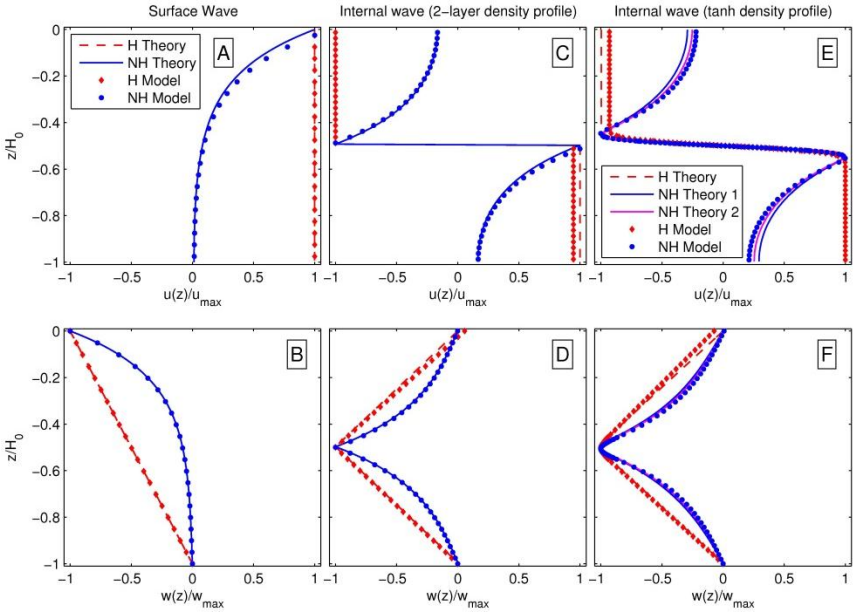
Dispersive properties





Horizontal and vertical velocity profil

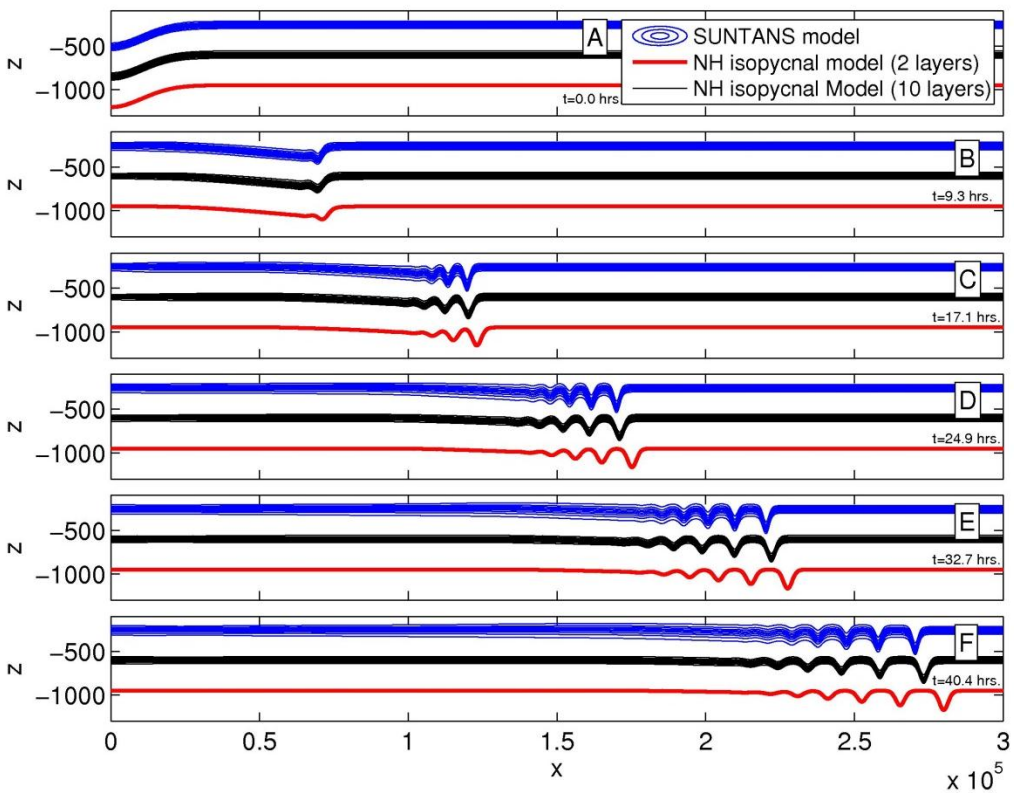




Test Cases

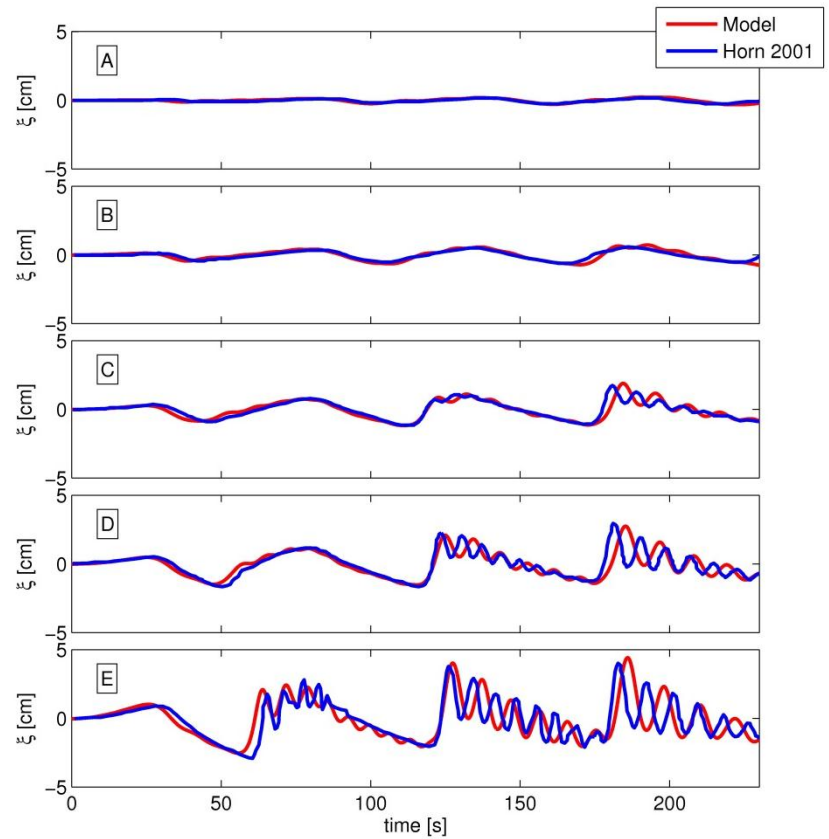
Formation of solitary waves:

Compared to other popular models



Compared to the SUNTANS z-level model after Vitousek & Fringer (2011)

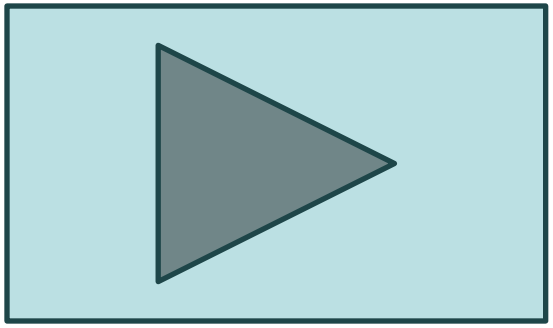
Compared to laboratory experiments



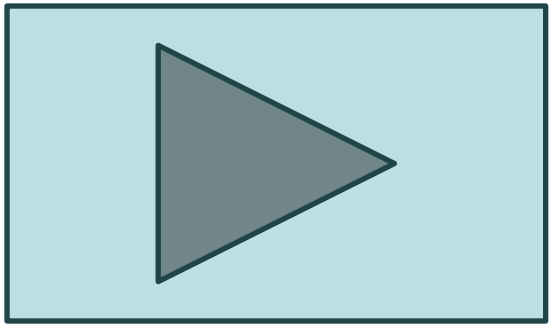
Compared to Horn et a. (2001)

Test Cases

Internal waves
in a 2-layer fluid



Internal wave (beams) in a
continuously-stratified fluid



Conclusions

Nonhydrostatic models are **computationally expensive** !

Nonhydrostatic models require very high horizontal resolution ($\Delta x < h_1$).

Satisfaction of this condition ($\Delta x < h_1$) is a **significant** additional resolution
requirement beyond the current state-of-the-art in ocean modeling.

To cope with this strict resolution requirement, it becomes necessary to find
means of reducing resolution requirements and computational cost.

Isopycnal coordinates provide a natural representation of a stratified fluid
which in turn reduces the number of vertical grid points and thus improves
efficiency.

Acknowledgements

DOE CSGF

Thank You So Much!

Numerical Dispersion

KdV:

$$\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} - \frac{3}{2}\delta\xi\frac{\partial \xi}{\partial x} + \frac{\varepsilon^2}{6}\frac{\partial^3 \xi}{\partial x^3} = 0$$

$$\frac{\xi_{i+1}^n - \xi_{i-1}^n}{2\Delta x} = \left.\frac{\partial \xi}{\partial x}\right|_i^n + \boxed{\left.\frac{\Delta x^2}{6}\frac{\partial^3 \xi}{\partial x^3}\right|_i^n} + \left.\frac{\Delta x^4}{120}\frac{\partial^5 \xi}{\partial x^5}\right|_i^n + \left.\frac{\Delta x^6}{5040}\frac{\partial^7 \xi}{\partial x^7}\right|_i^n + O\left(\Delta x^8\right)$$

Error mimics physical dispersion

Modified Equivalent KdV:

$$\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} - \frac{3}{2}\delta\xi\frac{\partial \xi}{\partial x} + \boxed{\left(1+\Gamma\right)\frac{\varepsilon^2}{6}\frac{\partial^3 \xi}{\partial x^3}} = O\left(\Delta x^4, \Delta t^4, \delta\Delta x^2, \varepsilon^2\Delta x^2\right)$$

$$\Gamma = \frac{\text{numerical dispersion}}{\text{physical dispersion}} = K\left(\frac{\Delta x_{\text{non-dim}}}{\varepsilon}\right)^2 = K\left(\frac{\Delta x}{h_1}\right)^2 = K\lambda^2$$

Vitousek & Fringer (2011)

K is typically an $O(1)$ constant

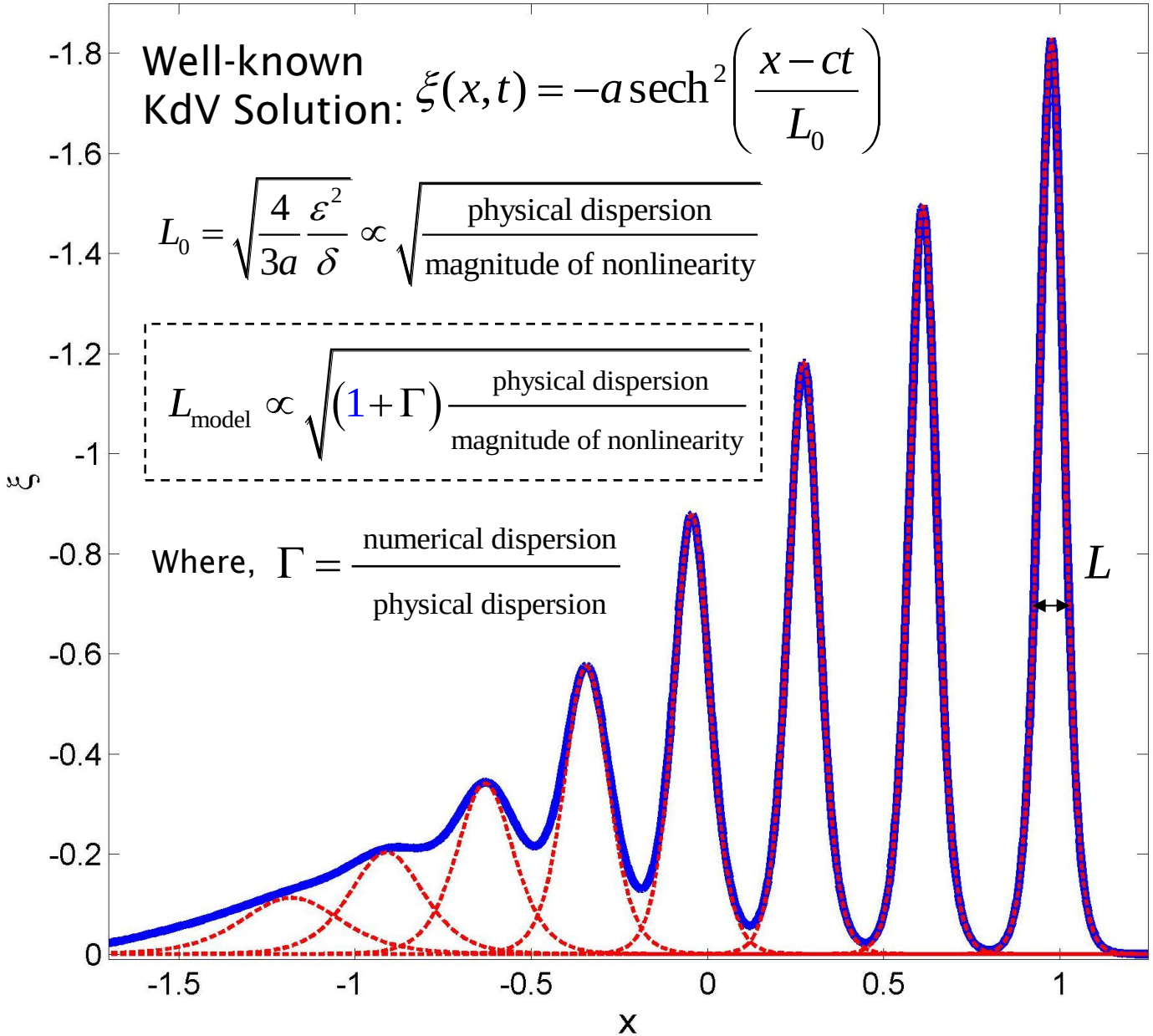
$\lambda \equiv \frac{\Delta x}{h_1}$ is the grid leptic ratio, or lepticity

Scotti & Mitran (2008)

Δx is the horizontal grid spacing

h_1 is the upper layer (pycnocline) depth

Widths of modeled solitary-like waves



Governing Equations
in *Continuous* Isopycnal Coordinates

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+fv=-\frac{1}{\rho_0}\frac{\partial p_h}{\partial x}-\frac{\partial q}{\partial x}+\frac{\partial}{\partial x}\left(\nu_{\mathrm{H}}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_{\mathrm{H}}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_{\mathrm{v}}\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}-fu=-\frac{1}{\rho_0}\frac{\partial p_h}{\partial y}-\frac{\partial q}{\partial y}+\frac{\partial}{\partial x}\left(\nu_{\mathrm{H}}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_{\mathrm{H}}\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_{\mathrm{v}}\frac{\partial v}{\partial z}\right)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial q}{\partial z}+\frac{\partial}{\partial x}\left(\nu_{\mathrm{H}}\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_{\mathrm{H}}\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_{\mathrm{v}}\frac{\partial w}{\partial z}\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

NH
RANS Eq.

$$\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\\ \frac{\partial}{\partial t}\end{bmatrix}=\begin{bmatrix}1 & 0 & -\frac{z_{x^*}}{z_\rho} & 0\\ 0 & 1 & -\frac{z_{y^*}}{z_\rho} & 0\\ 0 & 0 & \frac{1}{z_\rho} & 0\\ 0 & 0 & -\frac{z_{t^*}}{z_\rho} & 1\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x^*}\\ \frac{\partial}{\partial y^*}\\ \frac{\partial}{\partial \rho}\\ \frac{\partial}{\partial t^*}\end{bmatrix}$$

Transformation to
Isopycnal coordinates

Governing Equations
in *Continuous* Isopycnal Coordinates

$$\frac{\partial u}{\partial t^*} + u\frac{\partial u}{\partial x^*} + v\frac{\partial u}{\partial y^*} + fv = -\frac{\partial M}{\partial x^*} - \frac{\partial q}{\partial x^*} + \frac{\partial}{\partial x^*}\left(\nu_{\mathrm{H}}\frac{\partial u}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(\nu_{\mathrm{H}}\frac{\partial u}{\partial y^*}\right) + \frac{1}{z_\rho}\frac{\partial}{\partial \rho}\left(\nu_{\mathrm{v}}\frac{1}{z_\rho}\frac{\partial u}{\partial \rho}\right)$$

$$\frac{\partial v}{\partial t^*} + u\frac{\partial v}{\partial x^*} + v\frac{\partial v}{\partial y^*} - fu = -\frac{\partial M}{\partial y^*} - \frac{\partial q}{\partial y^*} + \frac{\partial}{\partial x^*}\left(\nu_{\mathrm{H}}\frac{\partial v}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(\nu_{\mathrm{H}}\frac{\partial v}{\partial y^*}\right) + \frac{1}{z_\rho}\frac{\partial}{\partial \rho}\left(\nu_{\mathrm{v}}\frac{1}{z_\rho}\frac{\partial v}{\partial \rho}\right)$$

$$\frac{\partial w}{\partial t^*} + u\frac{\partial w}{\partial x^*} + v\frac{\partial w}{\partial y^*} = -\frac{1}{z_\rho}\frac{\partial q}{\partial \rho} + \frac{\partial}{\partial x^*}\left(\nu_{\mathrm{H}}\frac{\partial w}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(\nu_{\mathrm{H}}\frac{\partial w}{\partial y^*}\right) + \frac{1}{z_\rho}\frac{\partial}{\partial \rho}\left(\nu_{\mathrm{v}}\frac{1}{z_\rho}\frac{\partial w}{\partial \rho}\right)$$

$$\frac{\partial}{\partial t^*}\left(z_\rho\right) + \frac{\partial}{\partial x^*}\left(z_\rho u\right) + \frac{\partial}{\partial y^*}\left(z_\rho v\right) = 0$$

$$\frac{\partial U}{\partial x^*} + \frac{\partial V}{\partial y^*} + \frac{\partial W}{\partial \rho} = 0$$

Where, $M = \frac{p_h}{\rho_0} + \frac{\rho g z}{\rho_0}$

$$U = z_\rho u$$
$$V = z_\rho v$$
$$W = w - z_{x^*}u - z_{y^*}v$$

Notes:

- Very similar to previous equations
- No vertical advection terms
- Ignore metric terms associated w/ NH pressure and viscosity

Model Equations
in *Discrete* Isopycnal (Layered) Coordinates

$$\frac{u_{i+1/2,k}^{n+1}-u_{i+1/2,k}^{n}}{\Delta t^*}=\frac{1}{2}(3+b)F_{i+1/2,k}^{n}-\frac{1}{2}(1+2b)F_{i+1/2,k}^{n-1}+\frac{b}{2}F_{i+1/2,k}^{n-2}$$
$$-\frac{1}{2}(1+c)g\frac{\eta_{i+1}^{n+1}-\eta_{i}^{n+1}}{\Delta x^*}-\frac{1}{2}(1-2c)g\frac{\eta_{i+1}^{n}-\eta_{i}^{n}}{\Delta x^*}-\frac{c}{2}g\frac{\eta_{i+1}^{n-1}-\eta_{i}^{n-1}}{\Delta x^*}$$
$$-\frac{(q_c)_{i+1,k}^{n+1/2}-(q_c)_{i,k}^{n+1/2}}{\Delta x^*}-\alpha\frac{q_{i+1,k}^{n-1/2}-q_{i,k}^{n-1/2}}{\Delta x^*}$$
$$+\theta\frac{(\tau^x)_{i+1/2,k-1/2}^{n+1}-(\tau^x)_{i+1/2,k+1/2}^{n+1}}{h_{i+1/2,k}^{n}}+(1-\theta)\frac{(\tau^x)_{i+1/2,k-1/2}^{n}-(\tau^x)_{i+1/2,k+1/2}^{n}}{h_{i+1/2,k}^{n}}$$

$$\frac{w_{i,k+1/2}^{n+1}-w_{i,k+1/2}^{n}}{\Delta t^*}=\frac{1}{2}(3+b)G_{i,k+1/2}^{n}-\frac{1}{2}(1+2b)G_{i,k+1/2}^{n-1}+\frac{b}{2}G_{i,k+1/2}^{n-2}$$
$$+\frac{(q_c)_{i,k+1}^{n+1/2}-(q_c)_{i,k}^{n+1/2}}{h_{i,k+1/2}^{n}}+\alpha\frac{q_{i,k+1}^{n-1/2}-q_{i,k}^{n-1/2}}{h_{i,k+1/2}^{n}}$$
$$+\theta\frac{(\tau^z)_{i,k}^{n+1}-(\tau^z)_{i,k+1}^{n+1}}{h_{i,k+1/2}^{n}}+(1-\theta)\frac{(\tau^z)_{i,k}^{n}-(\tau^z)_{i,k+1}^{n}}{h_{i,k+1/2}^{n}}$$

Where,

$$F_{i+1/2,k}^{n}=-C_H\left(u_{i+1/2,k}^{n}\right)+D_H\left(u_{i+1/2,k}^{n}\right)$$
$$-\frac{1}{\rho_0}\frac{(p_a)_{i+1,k}^{n}-(p_a)_{i,k}^{n}}{\Delta x^*}-\frac{\left(M^{(bc)}\right)_{i+1,k}^{n}-\left(M^{(bc)}\right)_{i,k}^{n}}{\Delta x^*}+fv_{i+1/2,k}^{n}$$
$$G_{i,k+1/2}^{n}=-C_H\left(w_{i,k+1/2}^{n}\right)+D_H\left(w_{i,k+1/2}^{n}\right)-fu_{i,k+1/2}^{n}$$

Notes:

- Second-Order accurate in time & space
- Staggered C-grid layout
- Split Montgomery potential into Barotropic (implicit) & Baroclinic (explicit) parts
- IMplicit-EXplicit (IMEX) multistep method. Durran (2012)
- θ - method for vertical diffusion
- Explicit horizontal diffusion
- Pressure projection (1st order)