Extending the Flux Operator with Coherent-State Projections

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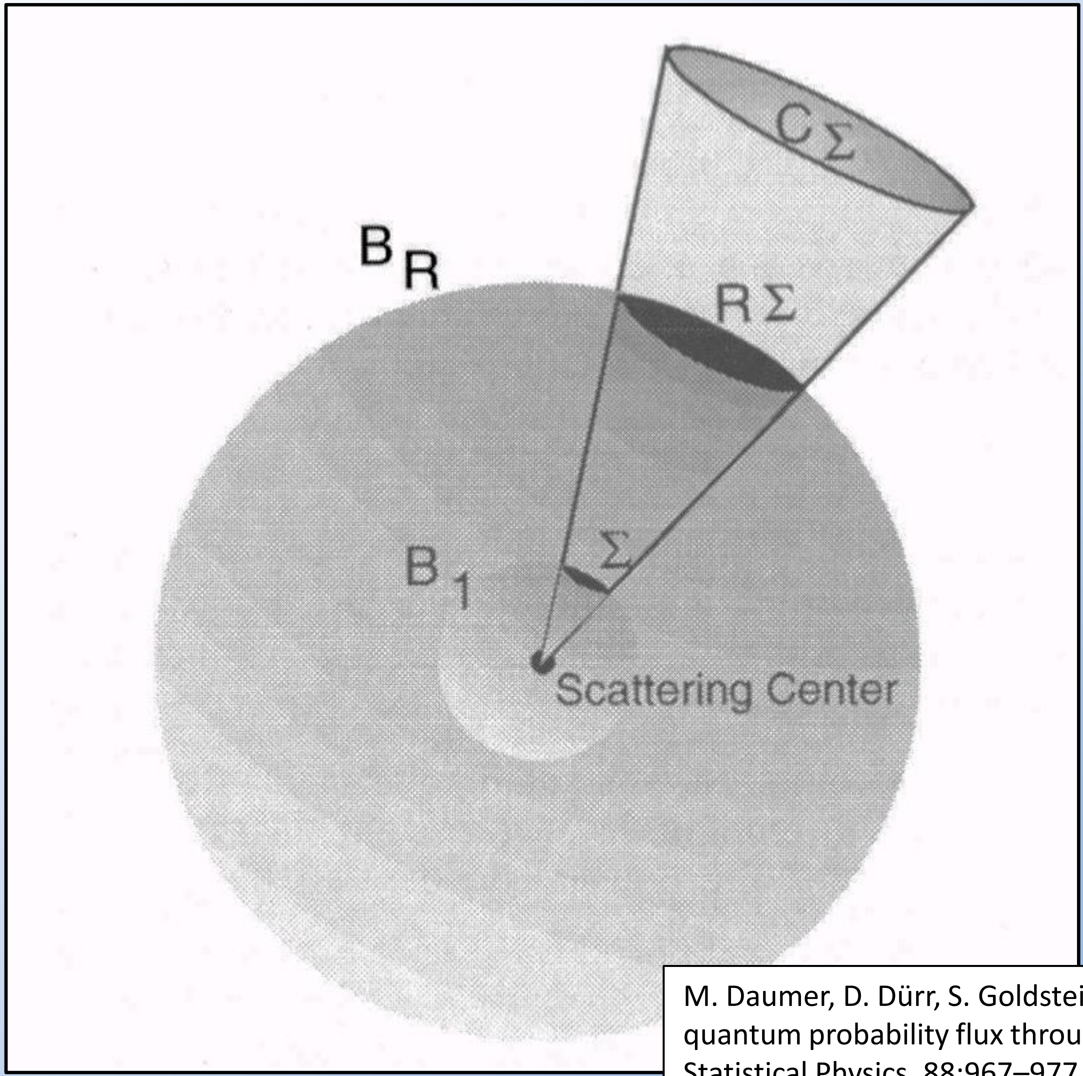
DOE CSGF Conference 2012

Washington, D.C.

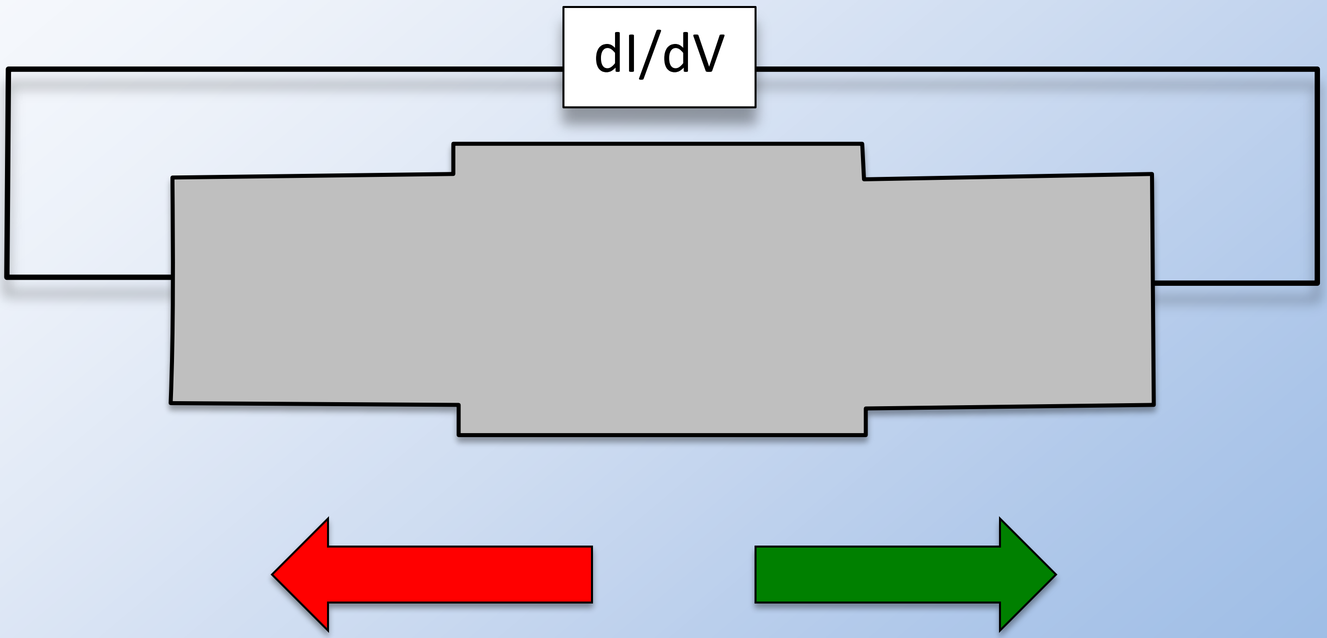
The Problem

Quantum Scattering



M. Daumer, D. Dürr, S. Goldstein, and N. Zanghi. On the quantum probability flux through surfaces. Journal of Statistical Physics, 88:967–977, 1997.

Quantum Scattering



Does the electron propagate or reflect back? We use the flux operator to find out.

The Flux Operator

$$\vec{j} = \frac{\hbar}{2m\,i}\left(\Psi^{*}\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^{*}\right) = \frac{\hbar}{m}\mathrm{Im}(\Psi^{*}\vec{\nabla}\Psi)$$

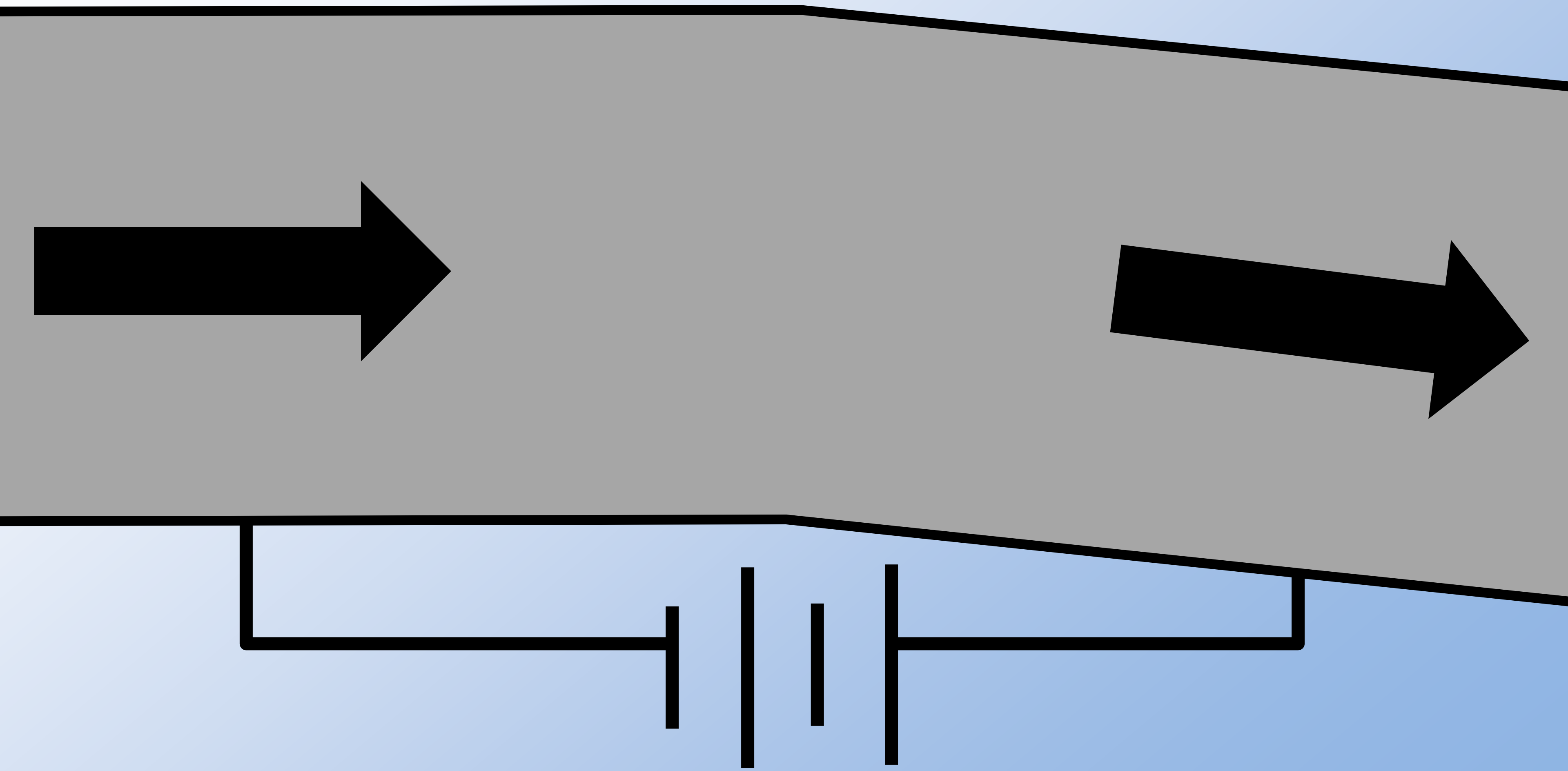
$$\hat{\mathbf{j}}_{\mathbf{r}} = \frac{1}{2m}\left(|\mathbf{r}\rangle\langle\mathbf{r}|\,\hat{\mathbf{p}} + \hat{\mathbf{p}}\,|\mathbf{r}\rangle\langle\mathbf{r}|\right)$$

The Flux Operator

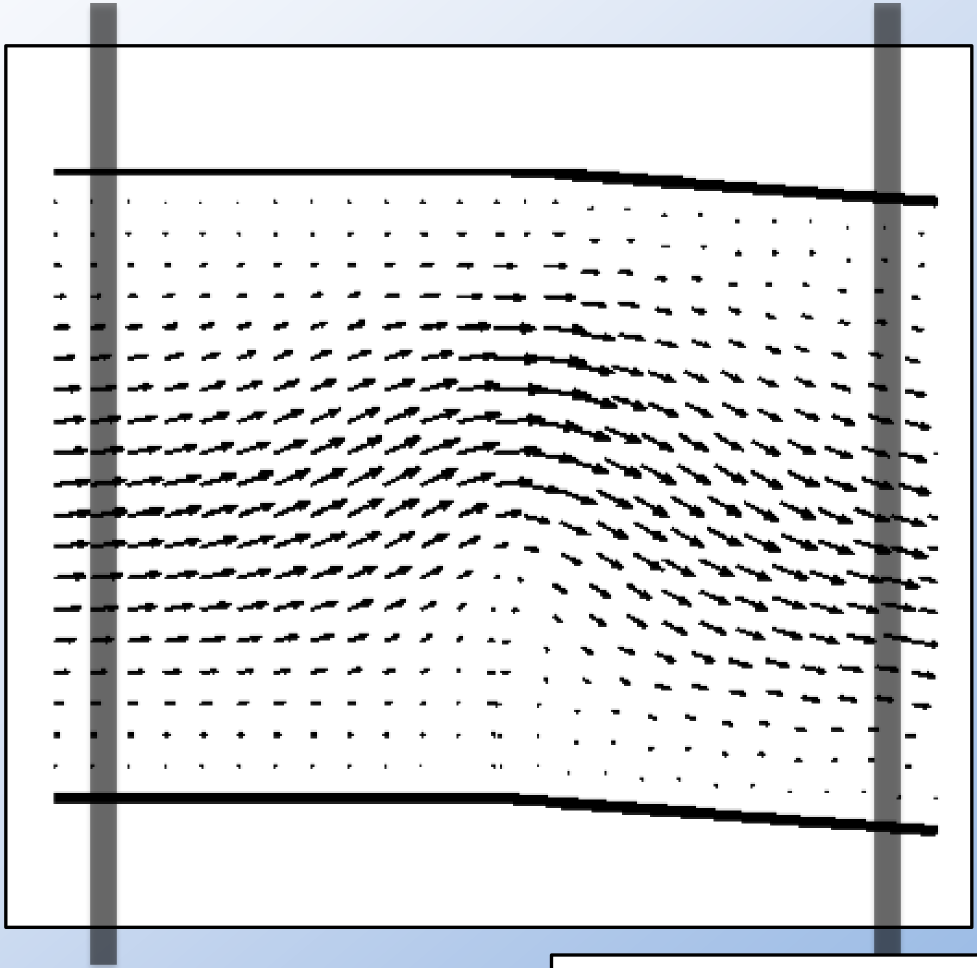
$$\Delta x \Delta k \geq \hbar/2$$

$$\hat{\mathbf{j}}_{\mathbf{r}} = \frac{1}{2m} \left(|\mathbf{r}\rangle \langle \mathbf{r}| \hat{\mathbf{p}} + \hat{\mathbf{p}} |\mathbf{r}\rangle \langle \mathbf{r}| \right)$$

Quantum Scattering

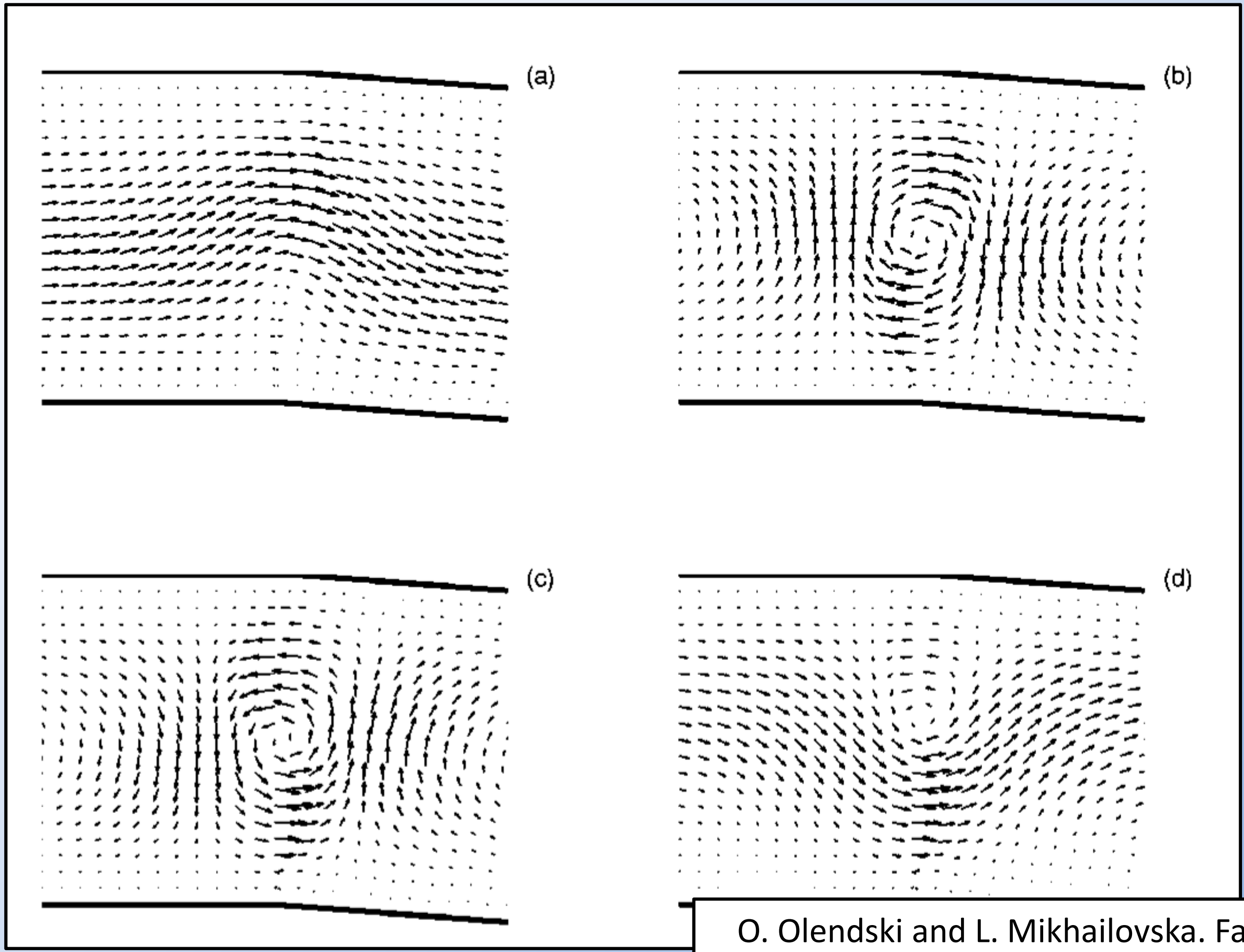


Quantum Scattering



O. Olendski and L. Mikhailovska. Fano resonances of a curved waveguide with an embedded quantum dot. *Phys. Rev. B, 67:035310, Jan 2003.*

Quantum Scattering



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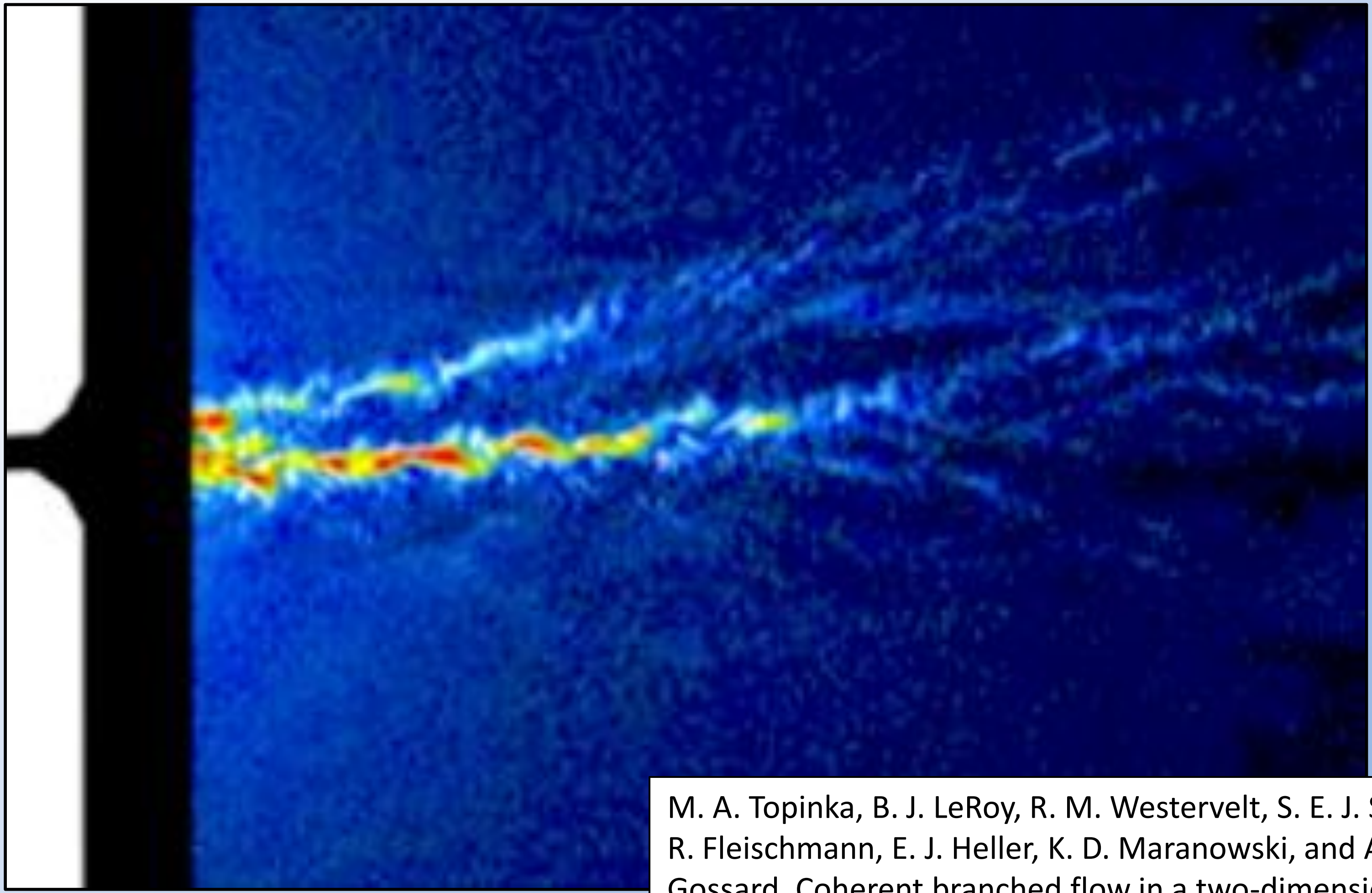

M. A. Topinka, B. J. LeRoy, R. M. Westervelt, S. E. J. Shaw, R. Fleischmann, E. J. Heller, K. D. Maranowski, and A. C. Gossard. Coherent branched flow in a two-dimensional electron gas. *Nature, 410(6825):183–186, 03 2001.*

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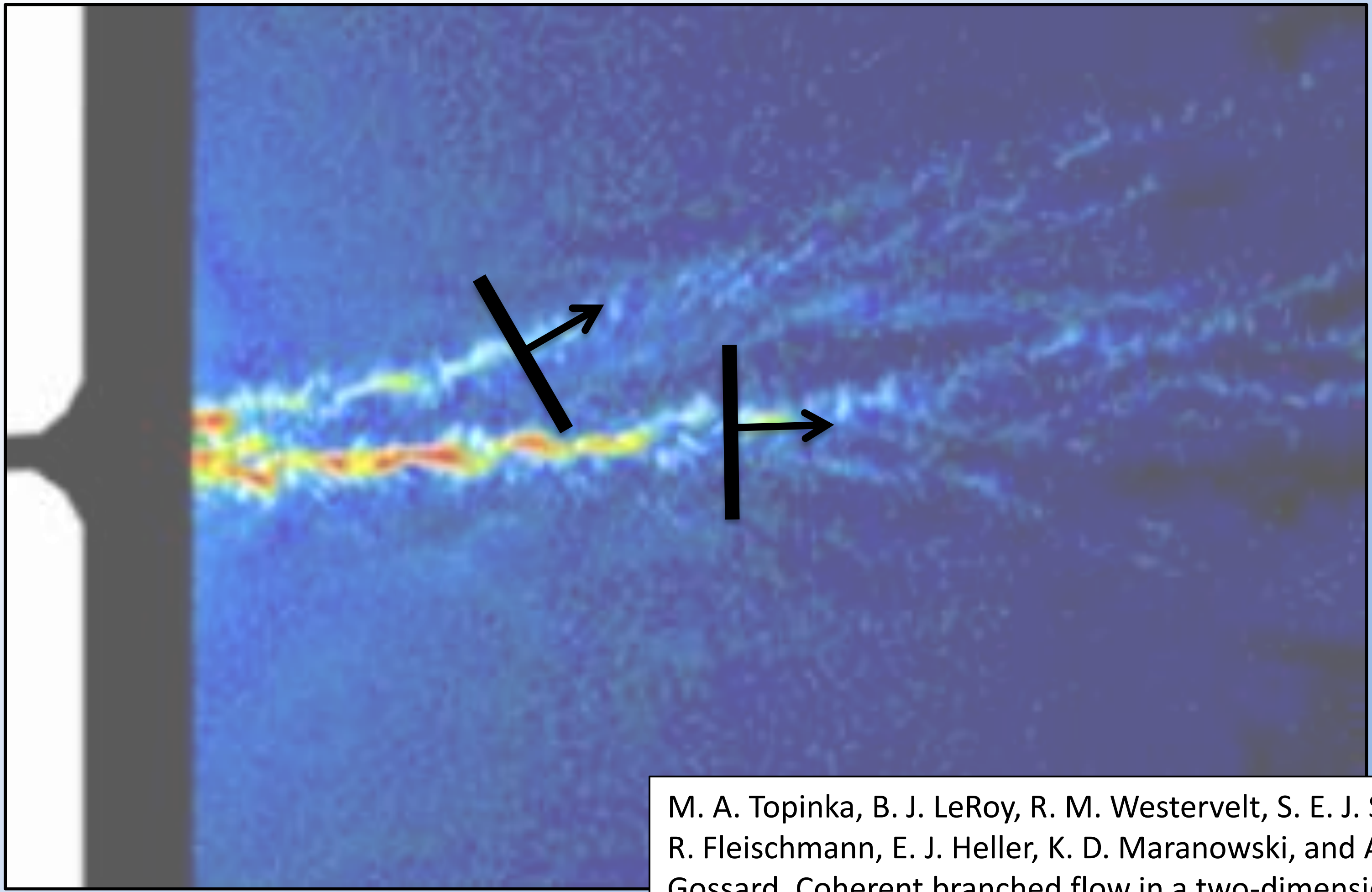

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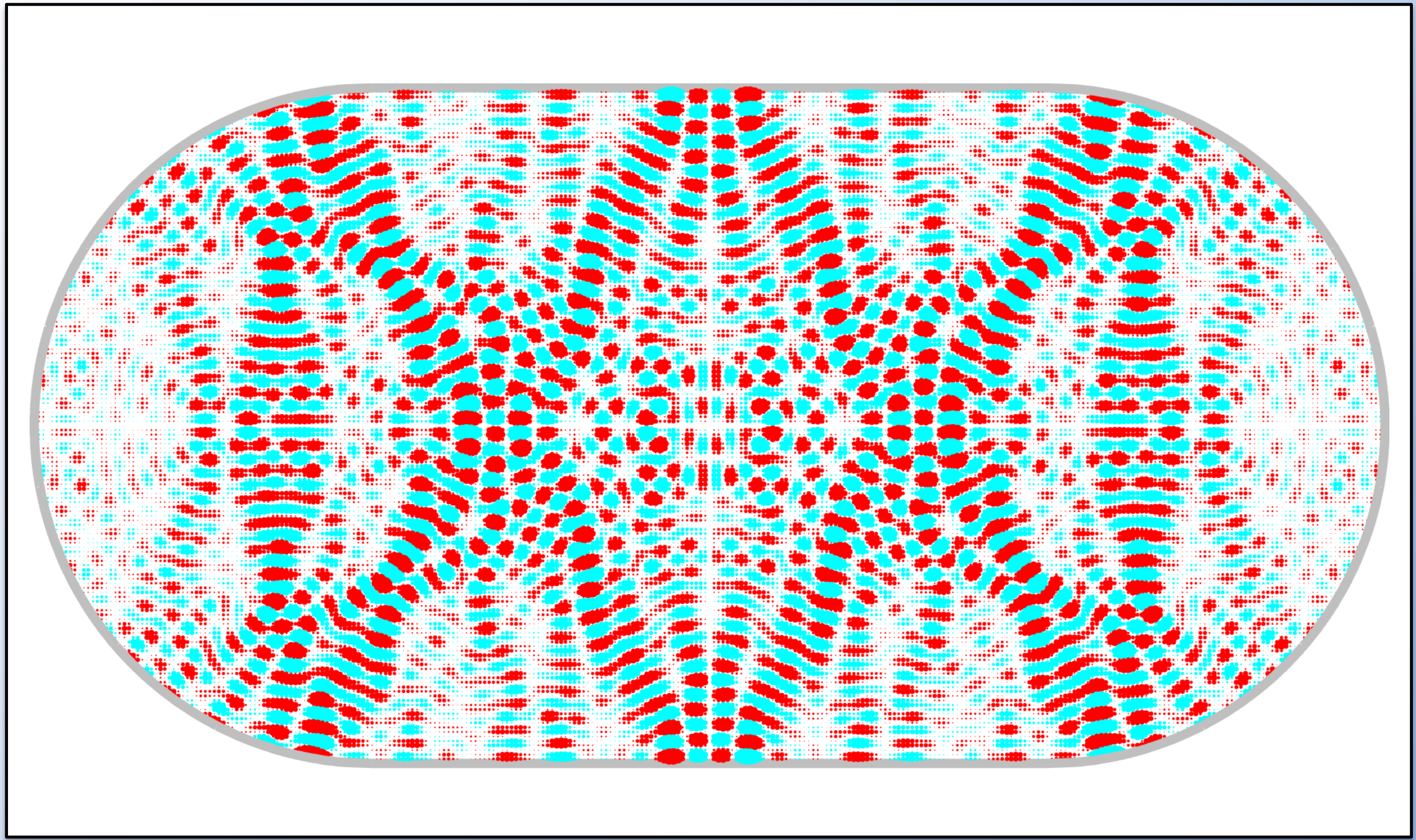
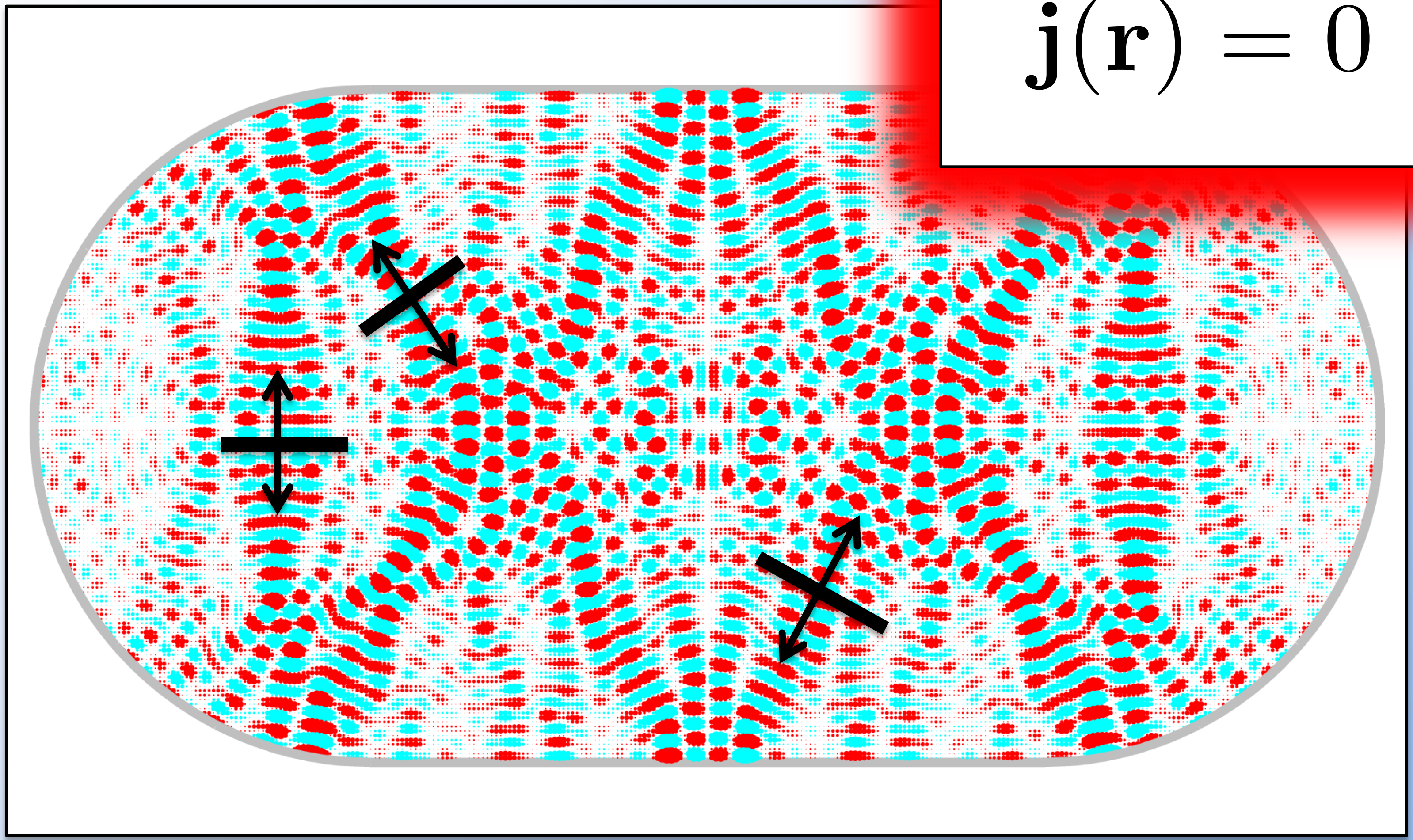
Resonance



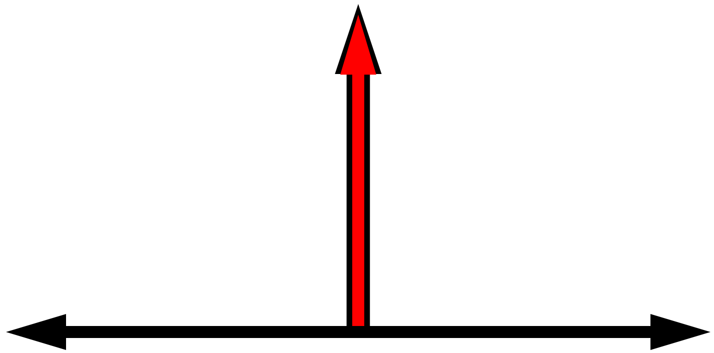
Resonance



$$\mathbf{j}(\mathbf{r}) = 0$$

Solution

$$\hat{\mathbf{j}}_{\mathbf{r}} = \frac{1}{2m}\left(|\mathbf{r}\rangle\langle\mathbf{r}|\,\hat{\mathbf{p}} + \hat{\mathbf{p}}\,|\mathbf{r}\rangle\langle\mathbf{r}|\right)$$



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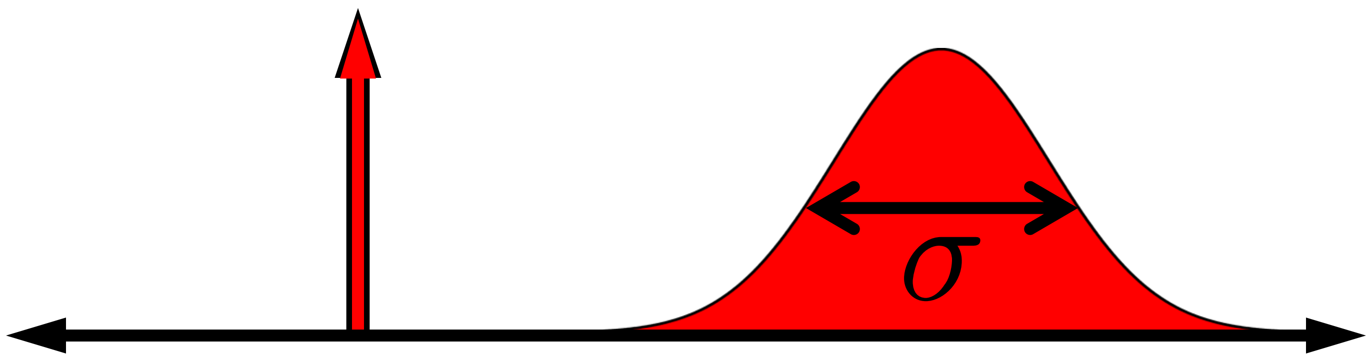
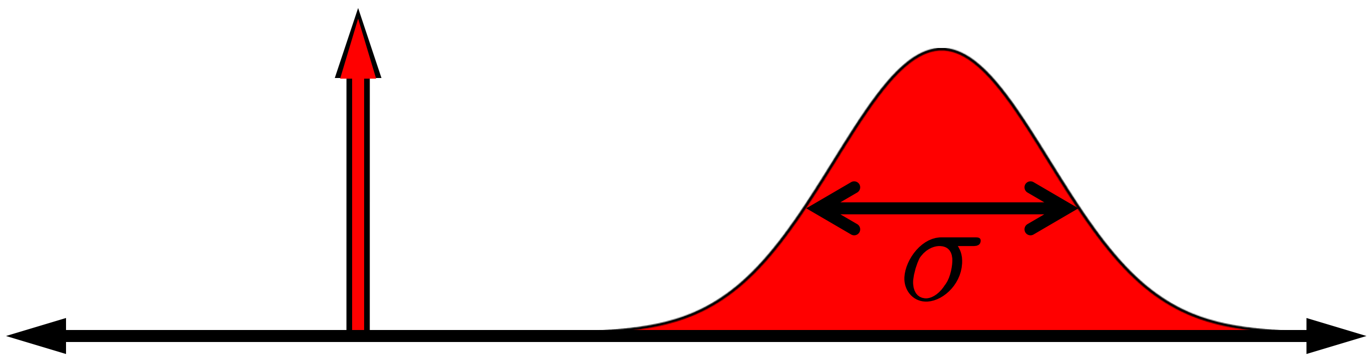


$$\langle\mathbf{r}|\,\mathbf{r}_0,\sigma\rangle = \left(\frac{1}{\sigma\sqrt{\pi/2}}\right)^{d/2} e^{-(\mathbf{r}-\mathbf{r}_0)^2/4\sigma^2}$$

$$\hat{\mathbf{j}}_{\mathbf{r}} = \frac{1}{2m}\left(|\mathbf{r}\rangle\langle\mathbf{r}|\,\hat{\mathbf{p}} + \hat{\mathbf{p}}\,|\mathbf{r}\rangle\langle\mathbf{r}|\right)$$



$$\langle\mathbf{r}|\,\mathbf{r}_0,\sigma\rangle = \left(\frac{1}{\sigma\sqrt{\pi/2}}\right)^{d/2} e^{-(\mathbf{r}-\mathbf{r}_0)^2/4\sigma^2}$$

$$\hat{\mathbf{j}}_{\mathbf{r}_0,\sigma} = \frac{1}{2m}\left(|\mathbf{r}_0,\sigma\rangle\langle\mathbf{r}_0,\sigma|\,\hat{\mathbf{p}} + \hat{\mathbf{p}}\,|\mathbf{r}_0,\sigma\rangle\langle\mathbf{r}_0,\sigma|\right)$$

$$\hat{\mathbf{j}}_{\mathbf{r}_0,\sigma} = \frac{1}{2m}\left(|\mathbf{r}_0,\sigma\rangle\langle\mathbf{r}_0,\sigma|\,\hat{\mathbf{p}} + \hat{\mathbf{p}}\,|\mathbf{r}_0,\sigma\rangle\langle\mathbf{r}_0,\sigma|\right)$$

Eigenvalue Eqn.:

$$\hat{j}_{\mathbf{r}_0,\sigma,i}\,|\lambda_{\sigma,i}\rangle = \lambda_{\sigma,i}\,|\lambda_{\sigma,i}\rangle$$

$$\hat{\mathbf{j}}_{\mathbf{r}_0,\sigma} = \frac{1}{2m}\left(|\mathbf{r}_0,\sigma\rangle\langle\mathbf{r}_0,\sigma|\,\hat{\mathbf{p}} + \hat{\mathbf{p}}\,|\mathbf{r}_0,\sigma\rangle\langle\mathbf{r}_0,\sigma|\right)$$

Eigenvalue Eqn.:

$$\hat{j}_{\mathbf{r}_0,\sigma,i}\,|\lambda_{\sigma,i}\rangle = \lambda_{\sigma,i}\,|\lambda_{\sigma,i}\rangle$$

Proposed Sol'n:

$$|\lambda_{\sigma,i}\rangle = |\mathbf{r}_0,\sigma\rangle + a\hat{p}_i\,|\mathbf{r}_0,\sigma\rangle$$

$$\hat{\mathbf{j}}_{\mathbf{r}_0,\sigma} = \frac{1}{2m}\left(\left|\mathbf{r}_0,\sigma\right\rangle\left\langle\mathbf{r}_0,\sigma\right|\hat{\mathbf{p}} + \hat{\mathbf{p}}\left|\mathbf{r}_0,\sigma\right\rangle\left\langle\mathbf{r}_0,\sigma\right|\right)$$

Eigenvalue Eqn.:

$$\hat{j}_{\mathbf{r}_0,\sigma,i}\left|\lambda_{\sigma,i}\right\rangle = \lambda_{\sigma,i}\left|\lambda_{\sigma,i}\right\rangle$$

Proposed Sol'n:

$$\left|\lambda_{\sigma,i}\right\rangle = \left|\mathbf{r}_0,\sigma\right\rangle + a\hat{p}_i\left|\mathbf{r}_0,\sigma\right\rangle$$

$$\left\langle\mathbf{r}\right|\lambda_{\sigma,i,\pm}\rangle = \left\langle\mathbf{r}\right|\mathbf{r}_0,\sigma\rangle \pm \frac{i}{\sigma}\mathbf{e}_i\cdot(\mathbf{r}-\mathbf{r}_0)\left\langle\mathbf{r}\right|\mathbf{r}_0,\sigma\rangle$$

$$\hat{\mathbf{j}}_{\mathbf{r}_0,\sigma} = \frac{1}{2m}\left(|\mathbf{r}_0,\sigma\rangle\langle\mathbf{r}_0,\sigma|\,\hat{\mathbf{p}} + \hat{\mathbf{p}}\,|\mathbf{r}_0,\sigma\rangle\langle\mathbf{r}_0,\sigma|\right)$$

Eigenvalue Eqn.:

$$\hat{j}_{\mathbf{r}_0,\sigma,i}\,|\lambda_{\sigma,i}\rangle = \lambda_{\sigma,i}\,|\lambda_{\sigma,i}\rangle$$

Proposed Sol'n:

$$|\lambda_{\sigma,i}\rangle = |\mathbf{r}_0,\sigma\rangle + a\hat{p}_i\,|\mathbf{r}_0,\sigma\rangle$$

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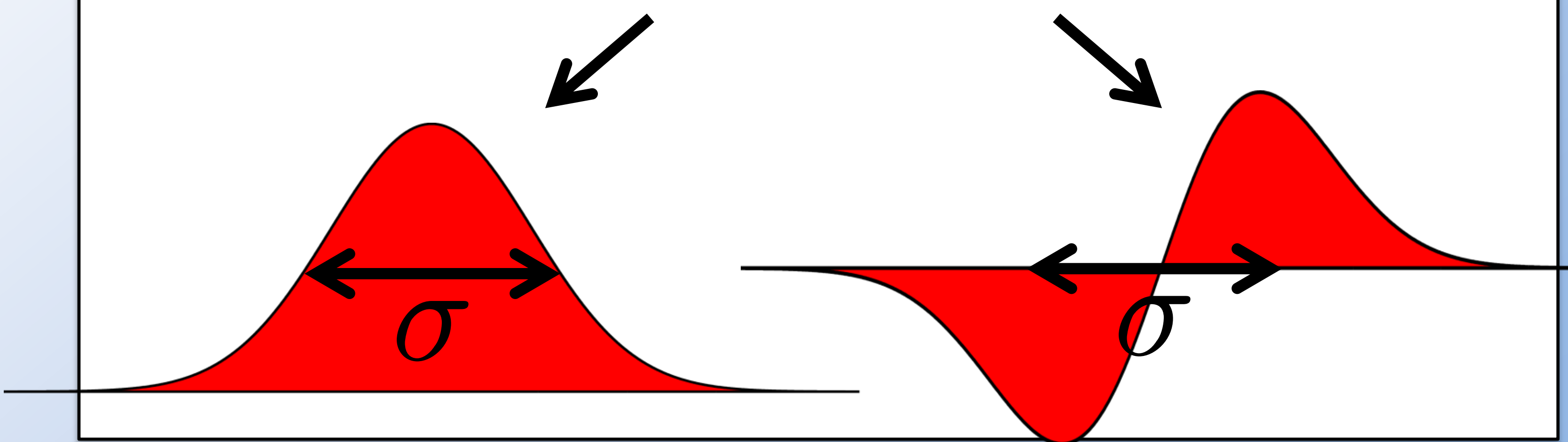
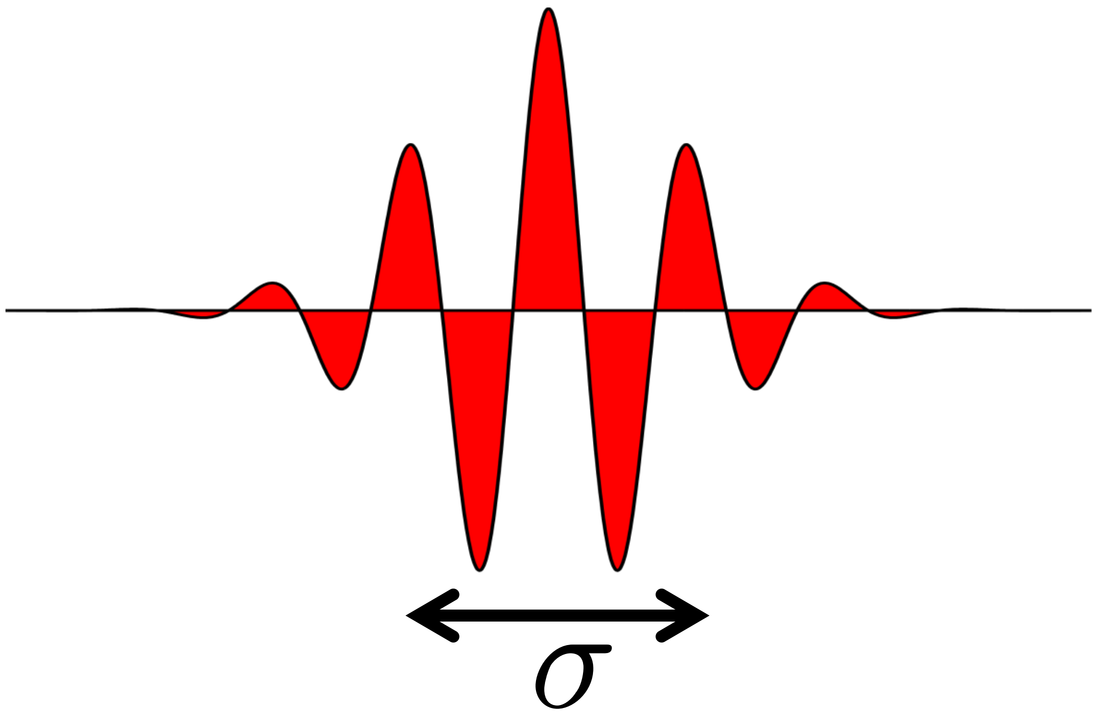


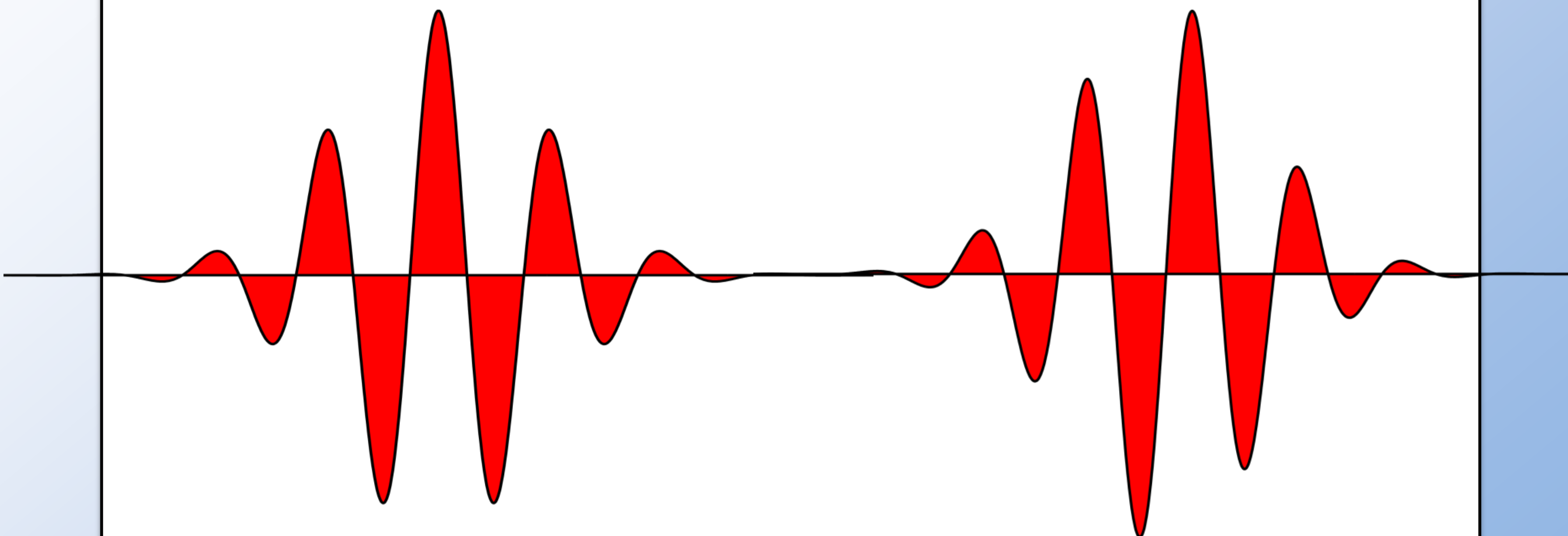
$$\mathrm{Re}\left[e^{-(\mathbf{r}-\mathbf{r}_0)^2/4\sigma^2+i\mathbf{k}_0\cdot\mathbf{r}_0}\right]$$



$$e^{-(\mathbf{r}-\mathbf{r}_0)^2/4\sigma^2+i\mathbf{k}_0\cdot\mathbf{r}_0}$$

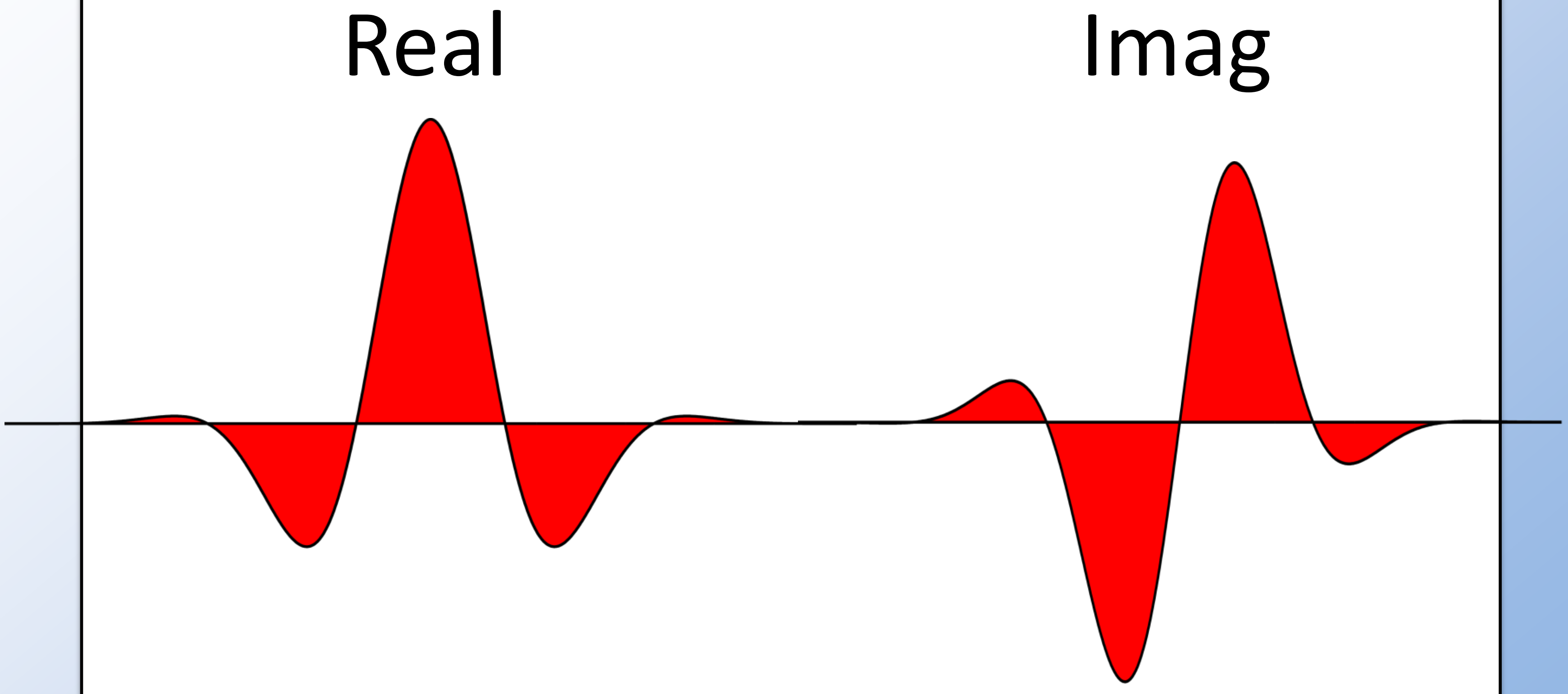
$$k_0\sigma = 8$$

Real Imag



$$e^{-(\mathbf{r}-\mathbf{r}_0)^2/4\sigma^2+i\mathbf{k}_0\cdot\mathbf{r}_0}$$

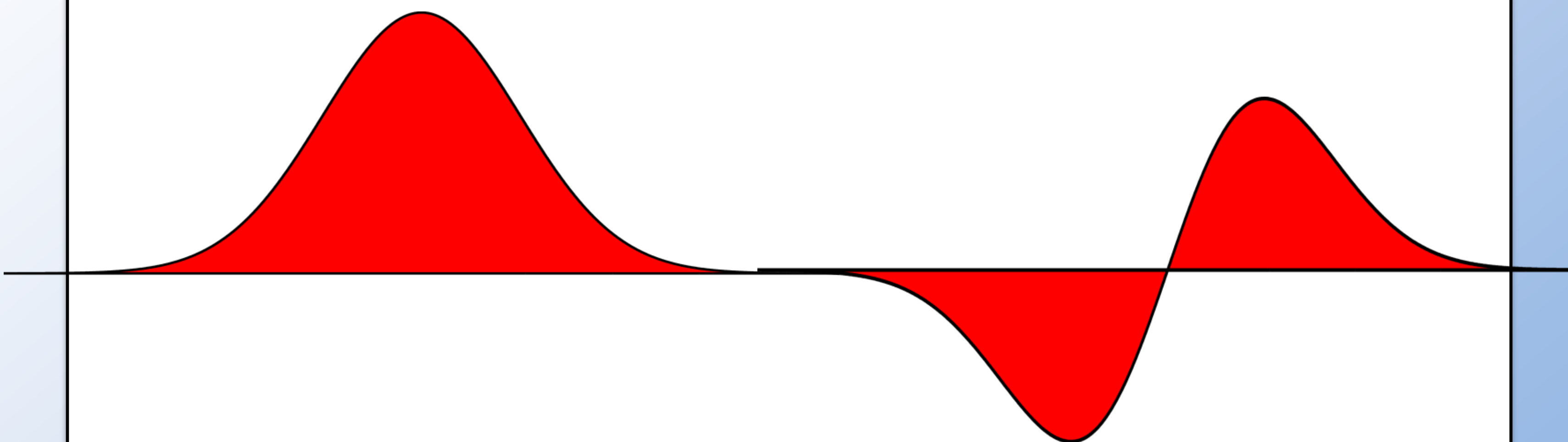
$$k_0\sigma = 3$$



$$e^{-(\mathbf{r}-\mathbf{r}_0)^2/4\sigma^2+i\mathbf{k}_0\cdot\mathbf{r}_0}$$

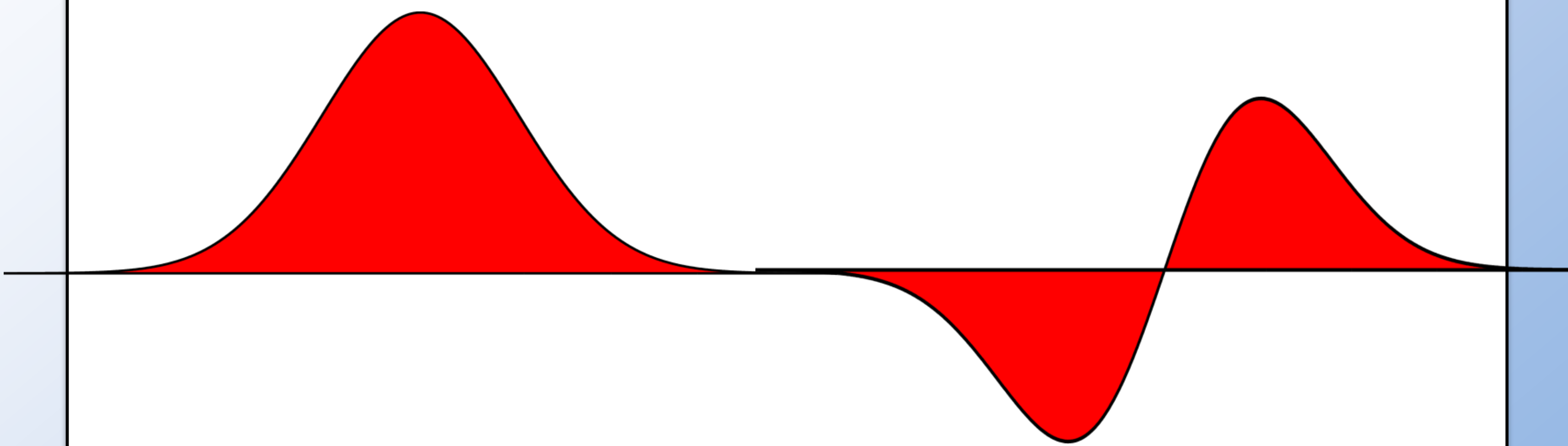
$$k_0\sigma = 1$$

Real Imag



Taylor Expansion of Coherent State

$$\langle \mathbf{r} | \, \mathbf{r}_0, \mathbf{k}_0, \sigma \rangle \approx \langle \mathbf{r} | \, \mathbf{r}_0, \sigma \rangle + i\mathbf{k}_0 \cdot (\mathbf{r} - \mathbf{r}_0) \langle \mathbf{r} | \, \mathbf{r}_0, \sigma \rangle$$



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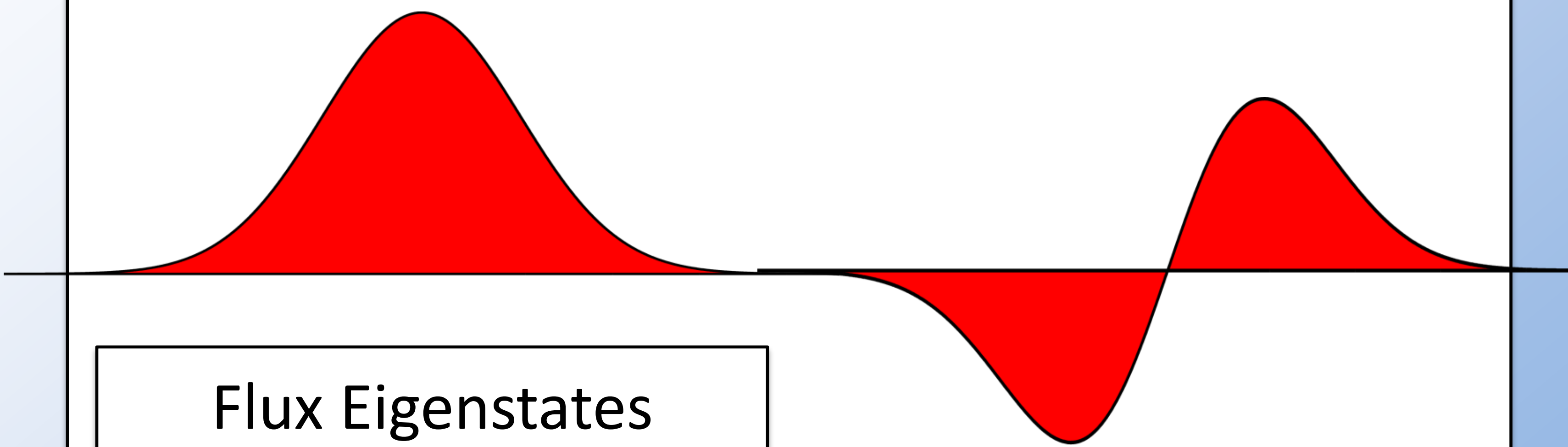


Flux Eigenstates

$$\langle \mathbf{r} | \lambda_{\sigma,i,\pm} \rangle = \langle \mathbf{r} | \mathbf{r}_0, \sigma \rangle \pm \frac{i}{\sigma} \mathbf{e}_i \cdot (\mathbf{r} - \mathbf{r}_0) \langle \mathbf{r} | \mathbf{r}_0, \sigma \rangle$$

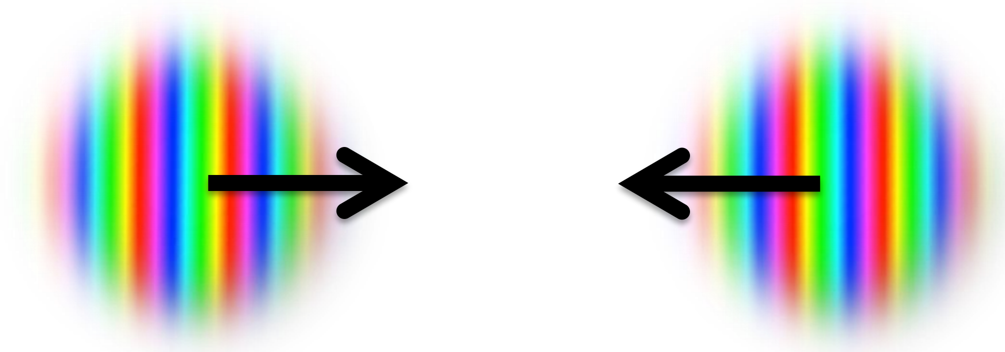
Flux expectation value:

$$\left\langle \psi \left| \hat{j}_{\mathbf{r}_0,\sigma,i} \right| \psi \right\rangle \quad = \quad \lambda \left|\langle \psi | \lambda_1 \rangle\right|^2 - \lambda \left|\langle \psi | \lambda_2 \rangle\right|^2$$

$$\Downarrow$$

$$\lim_{\sigma k_0 \to 0} \left\langle \psi \left| \hat{j}_{\mathbf{r}_0,\sigma,i} \right| \psi \right\rangle \quad =$$

$$\frac{\hbar k_0}{4 m \sigma^2} \left[\left|\langle \psi | \mathbf{r}_0, k_0 \mathbf{e}_i, \sigma \rangle\right|^2 - \left|\langle \psi | \mathbf{r}_0, -k_0 \mathbf{e}_i, \sigma \rangle\right|^2 \right]$$



Definition of the Husimi Function

$$\mathrm{Hu}(r_0, k_0, \sigma;\quad (r)) = |h\quad |r_0, k_0, \sigma i|^2$$

k_0

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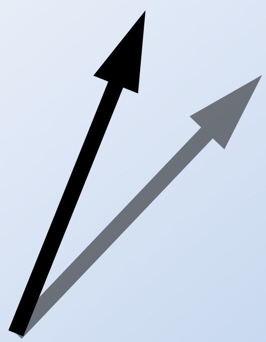
k_0



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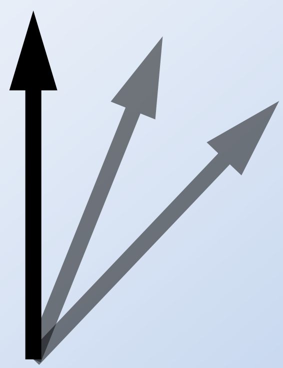
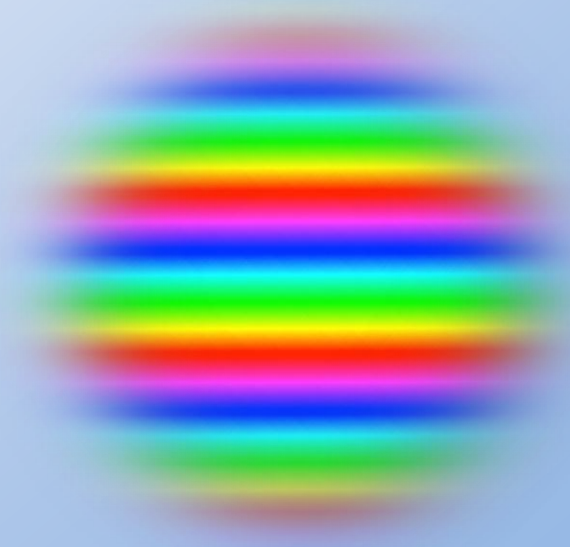
k_0

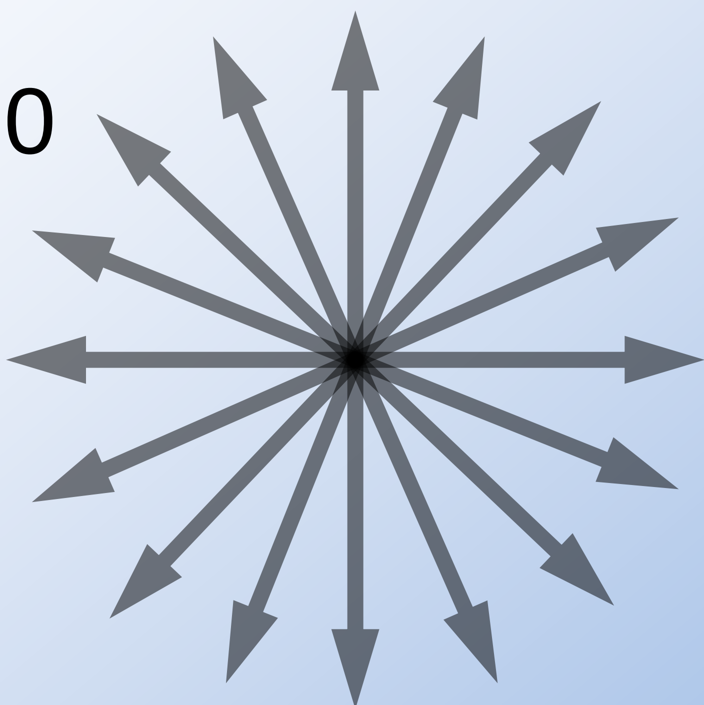




Definition of the Husimi Function

$$Hu(r_0, k_0, \sigma; \quad (r)) = |h \quad | r_0, k_0, \sigma i|^2$$

k_0



Definition of the Husimi Function

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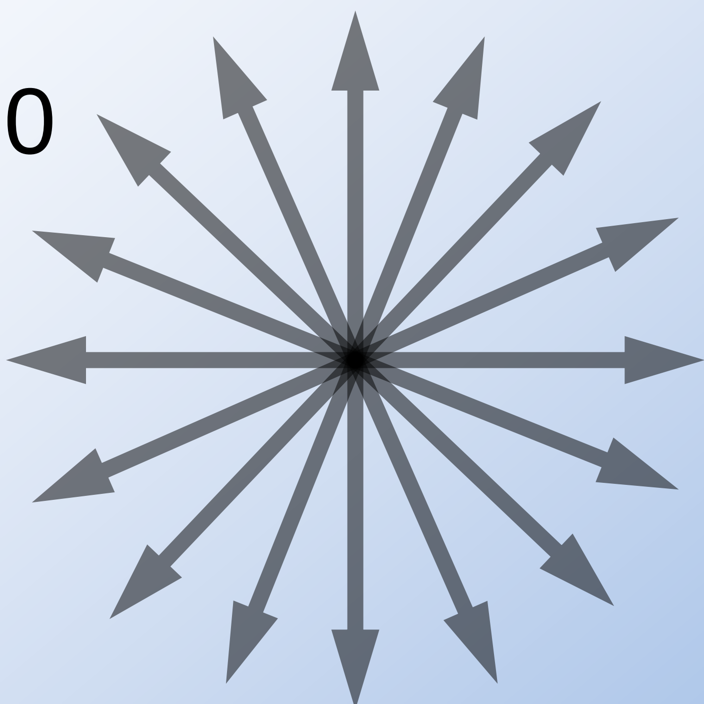
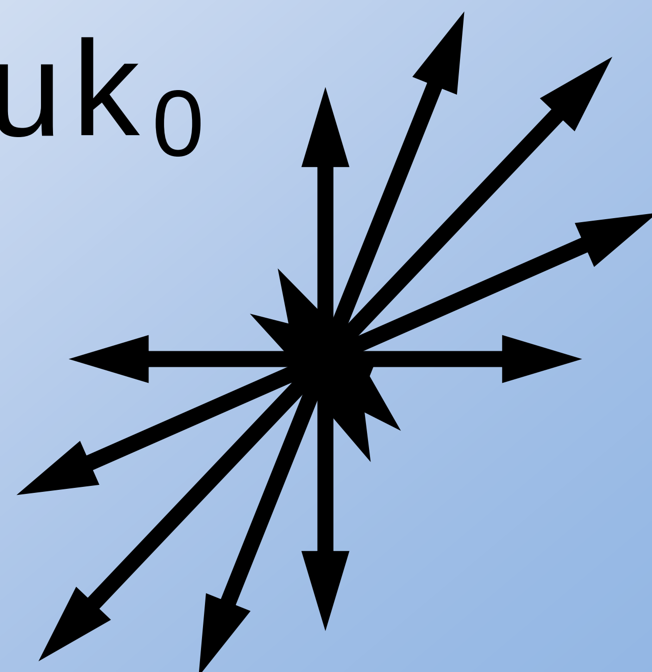
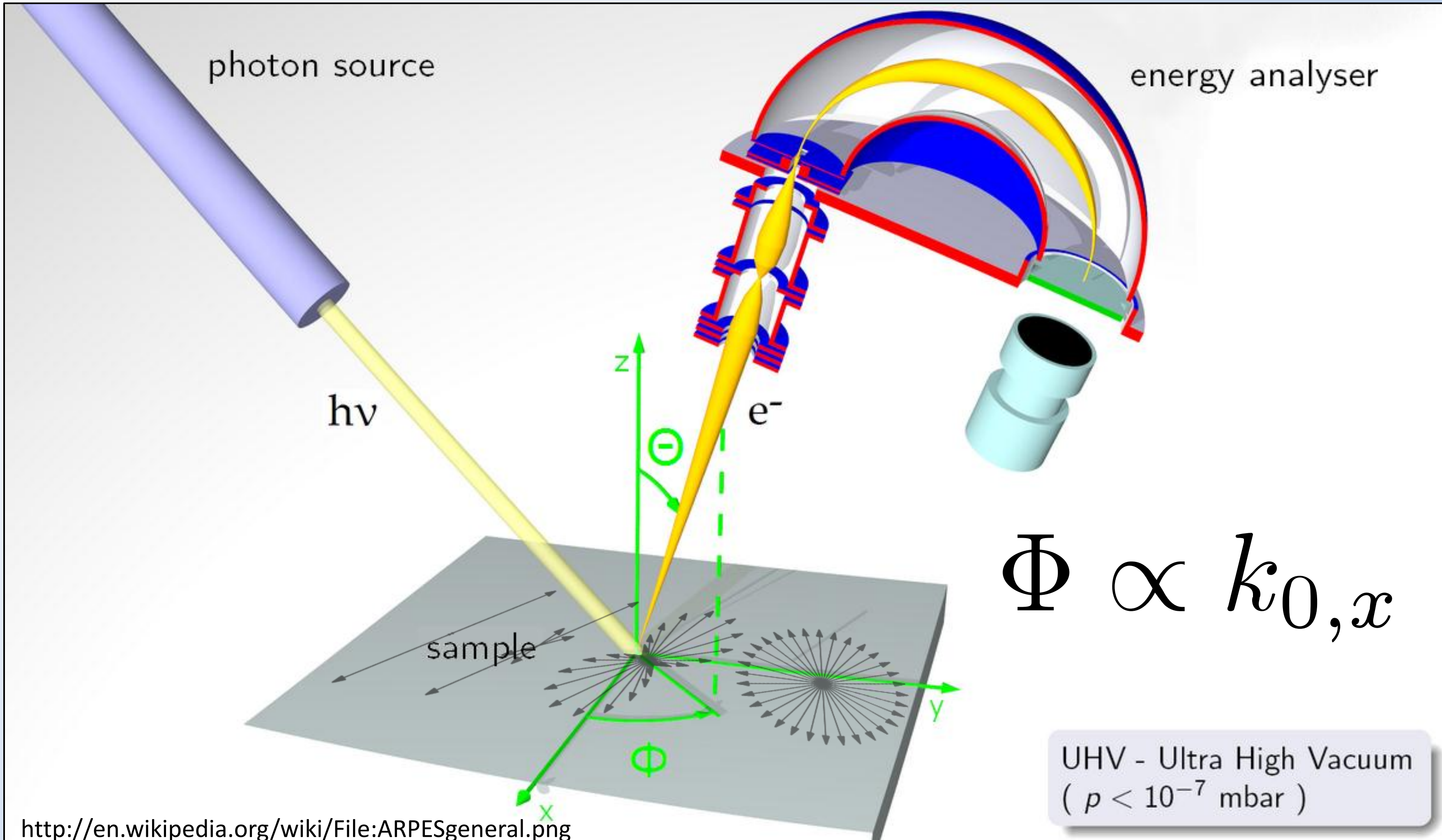


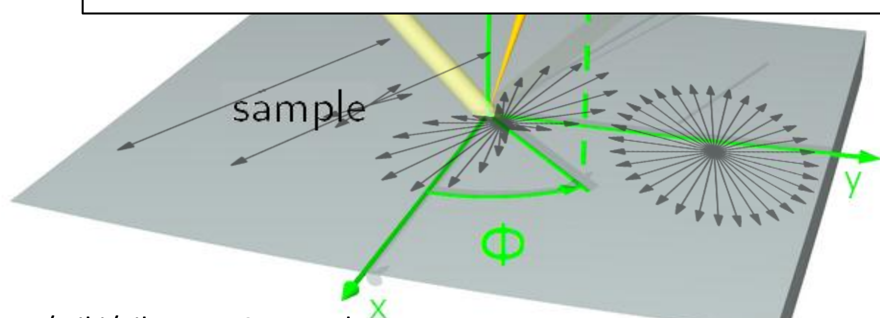


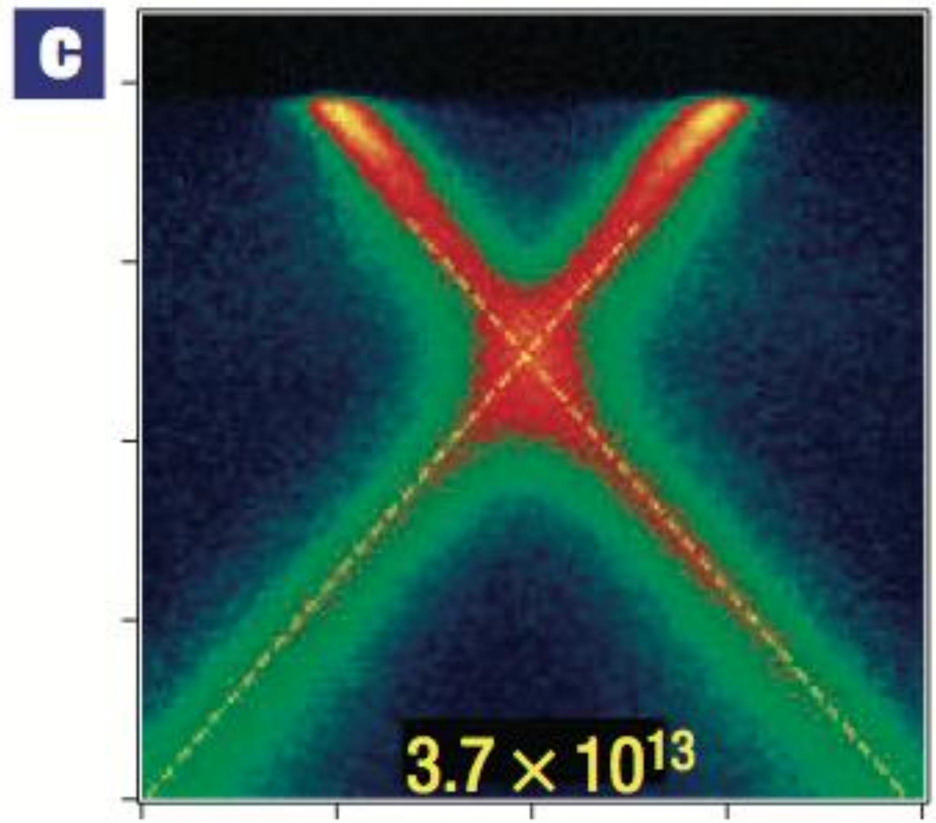
$$\left\langle \psi \left| \hat{\mathbf{j}}_{\mathbf{r}_0,\sigma} \right| \psi \right\rangle \approx \int \mathbf{k}_0 \left| \langle \psi | \, \mathbf{r}_0, \mathbf{k}_0, \sigma \rangle \right|^2 d^d k_0$$



$$\Phi \propto k_{0,x}$$

http://en.wikipedia.org/wiki/File:ARPESgeneral.png





A. Bostwick, T. Ohta, T. Seyller, K. Horn, and E. Rotenberg. Quasiparticle dynamics in graphene. *Nat Phys*, 3(1):36–40, 01 2007.

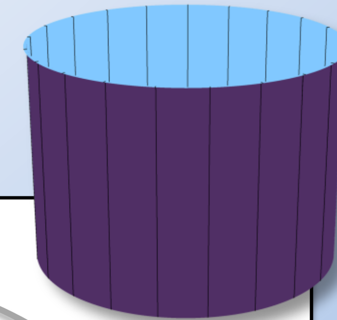
$$\Phi \propto k_{0,x}$$

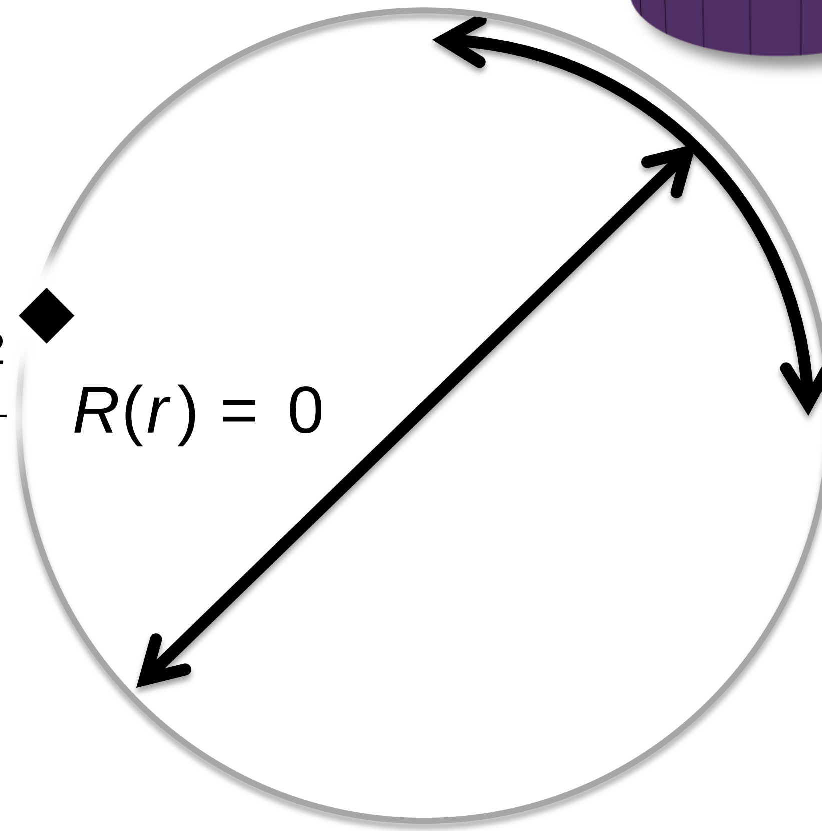
UHV - Ultra High Vacuum
($p < 10^{-7}$ mbar)

http://en.wikipedia.org/wiki/File:ARPESgeneral.png

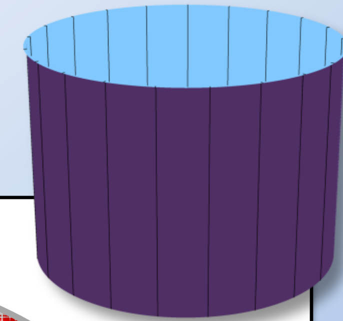
Examples

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(k^2 - \frac{m^2}{r^2}\right) R(r) = 0$$

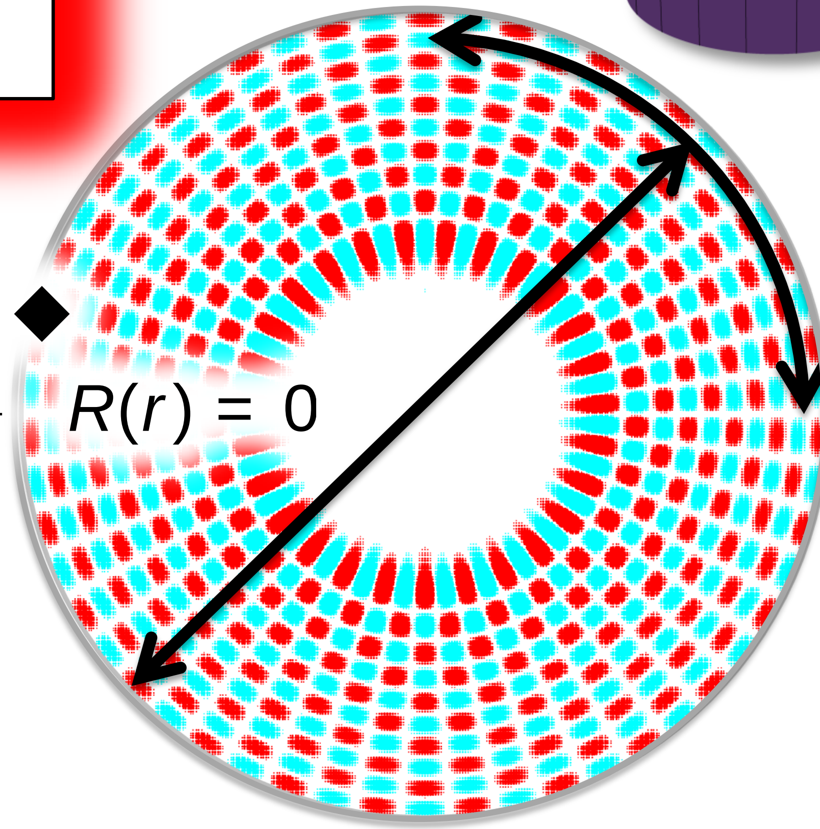


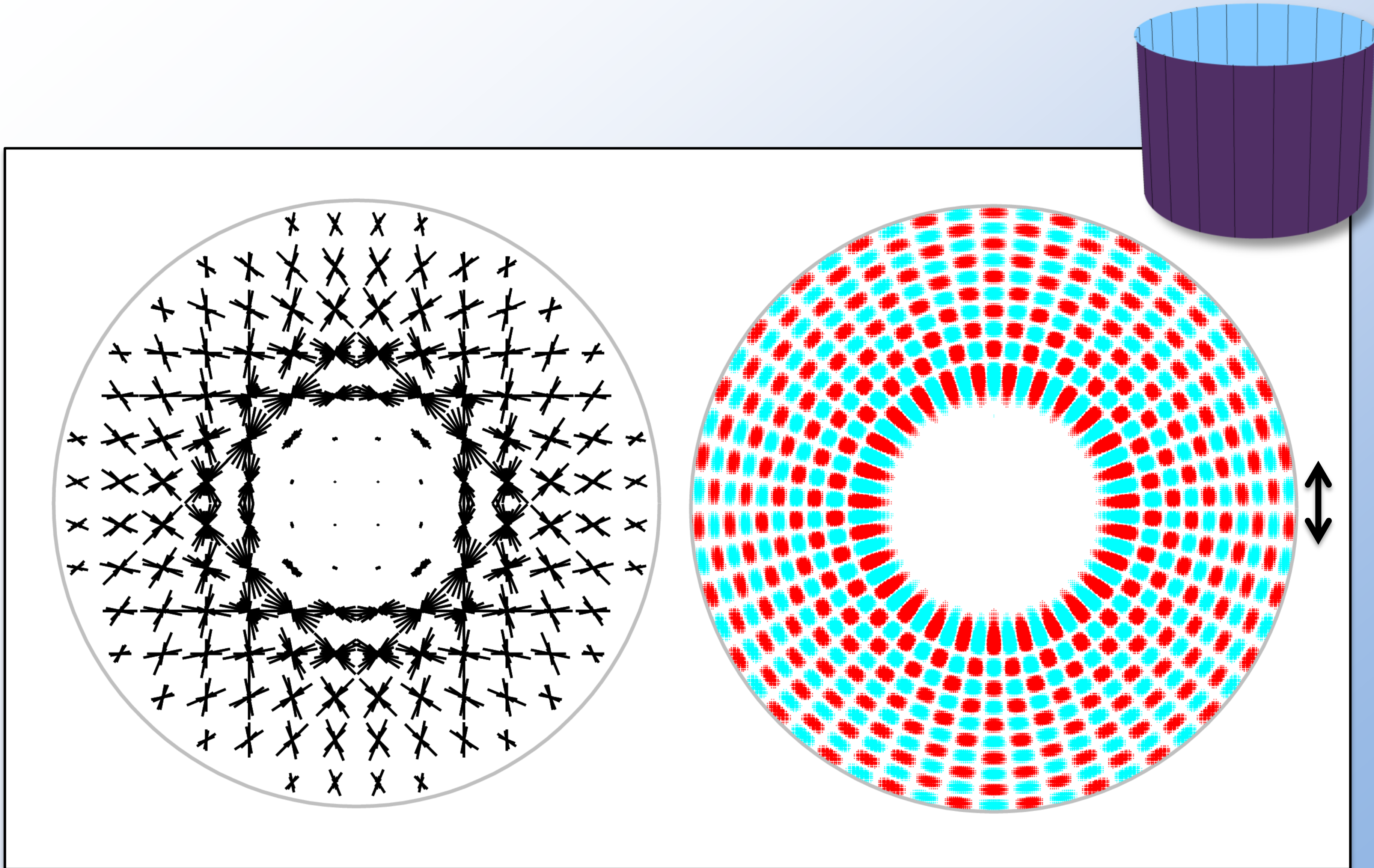


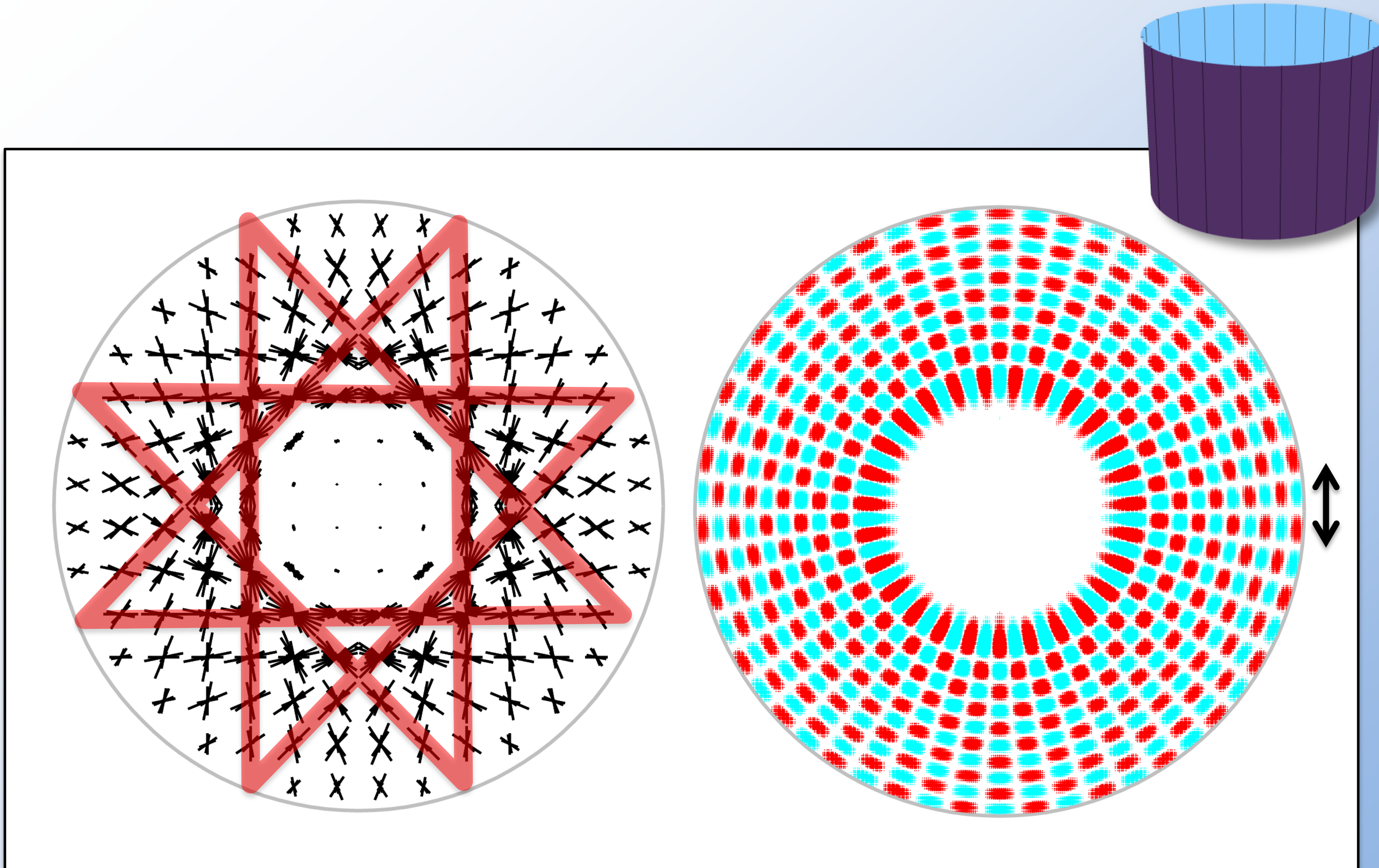
$$\boxed{\mathbf{j}(\mathbf{r}) = 0}$$

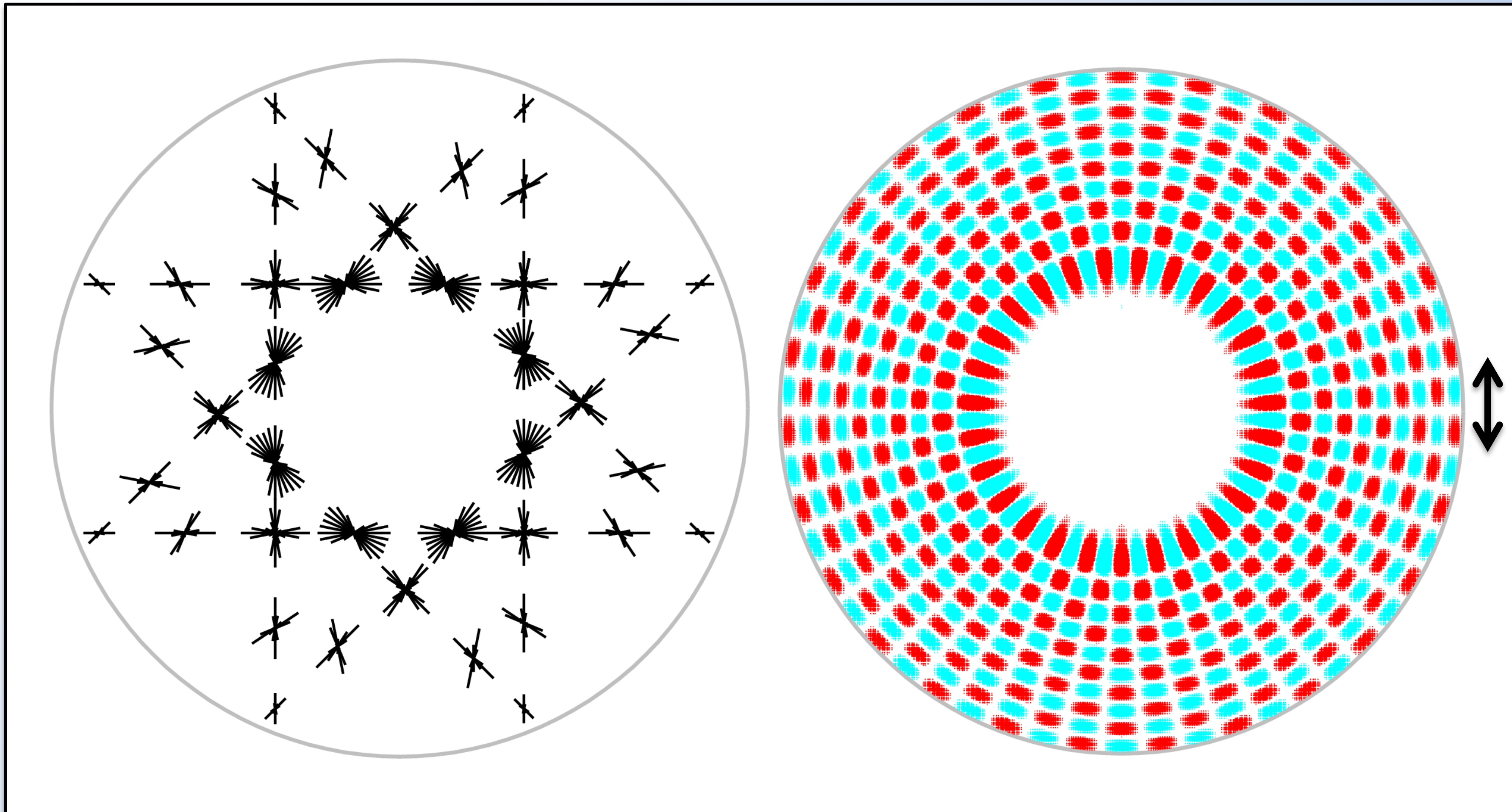


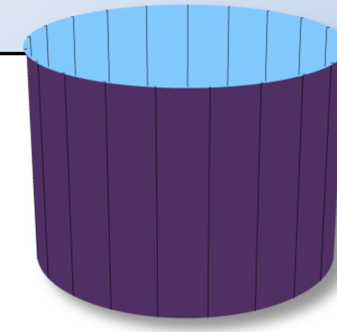
$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \checkmark\, k^2 - \frac{m^2}{r^2} \blacklozenge\, R(r) = 0$$

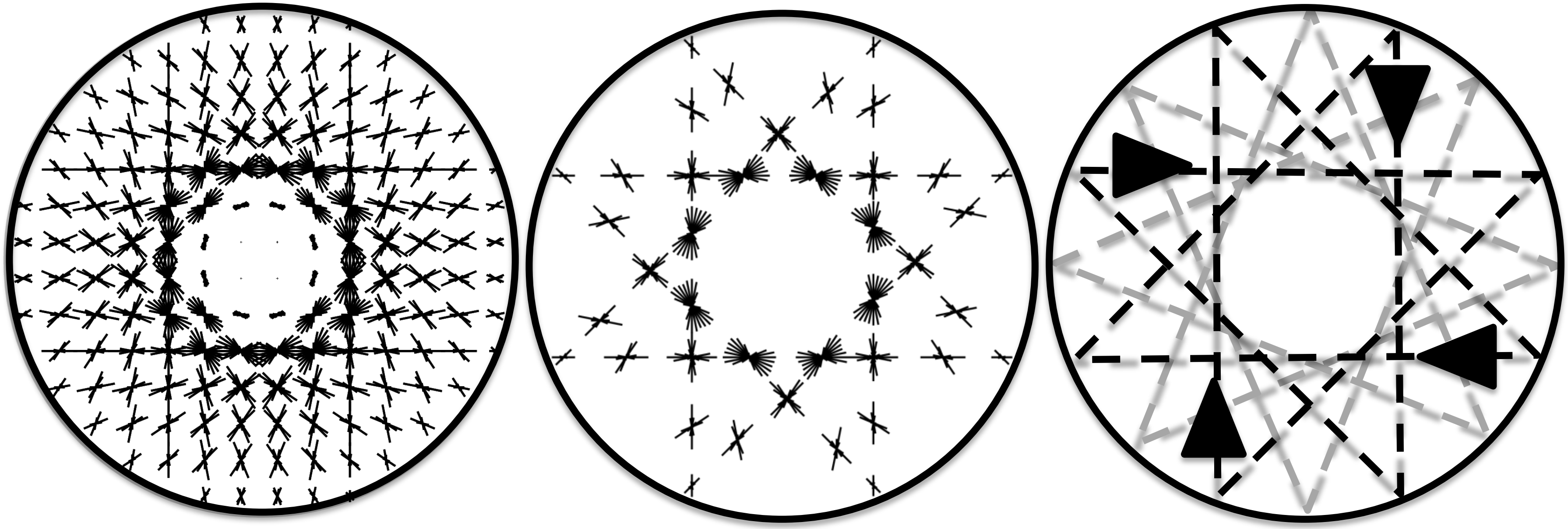


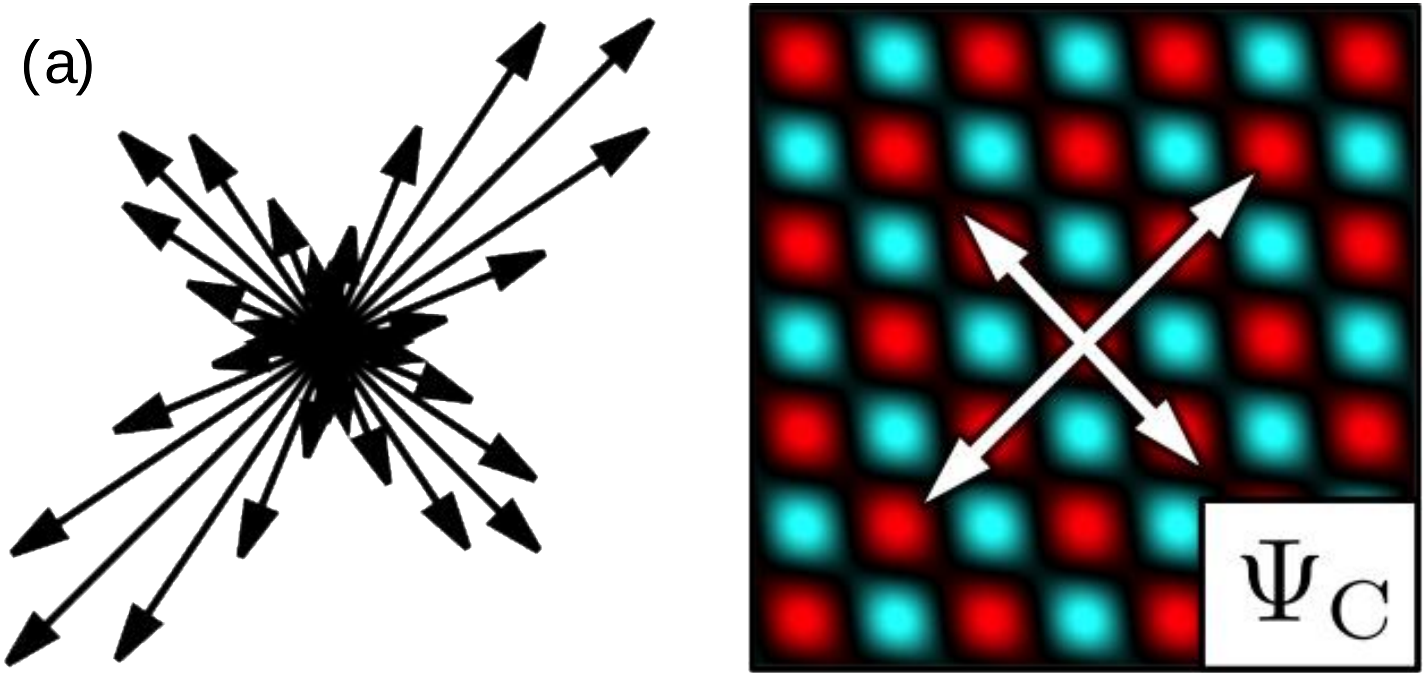




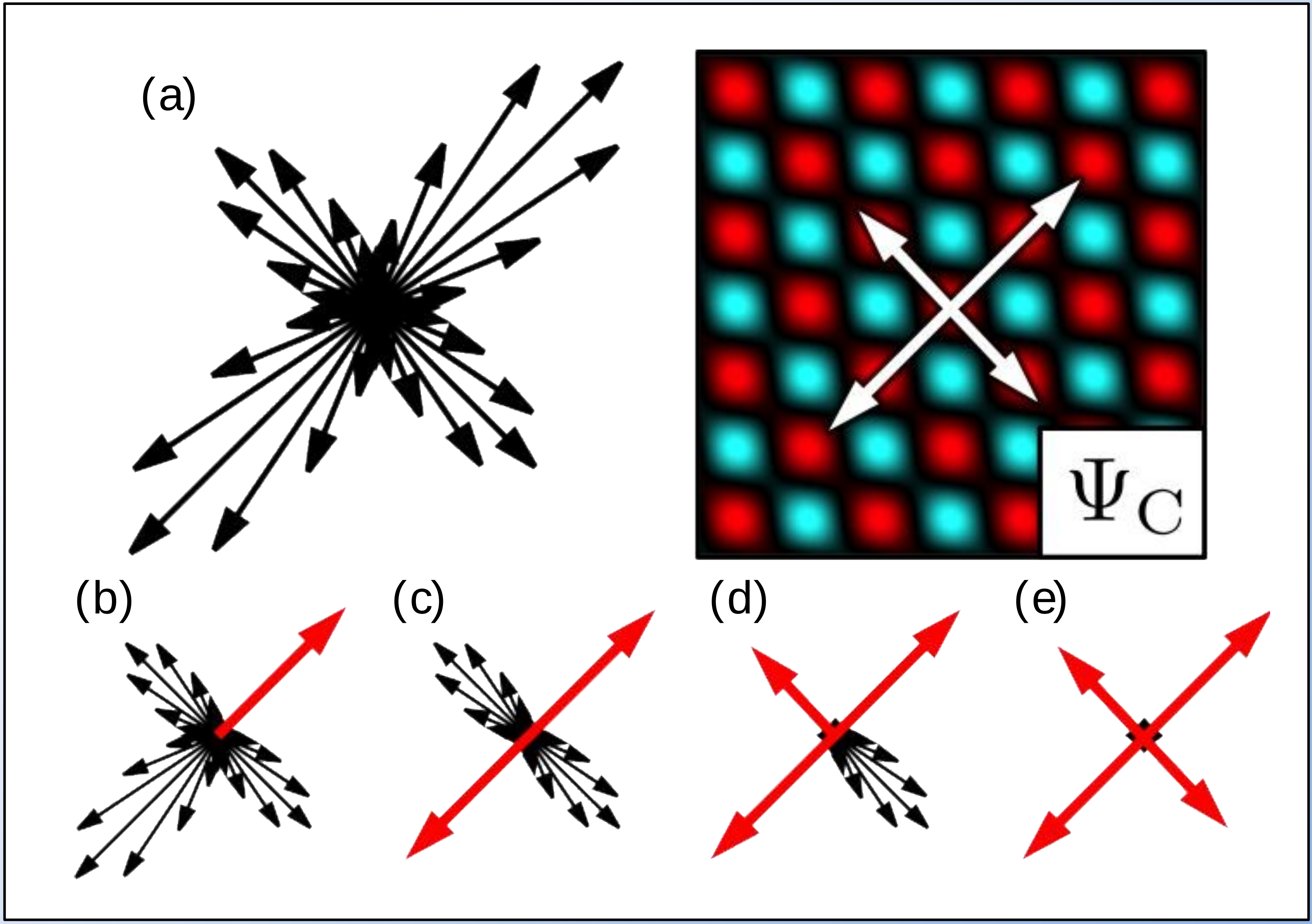




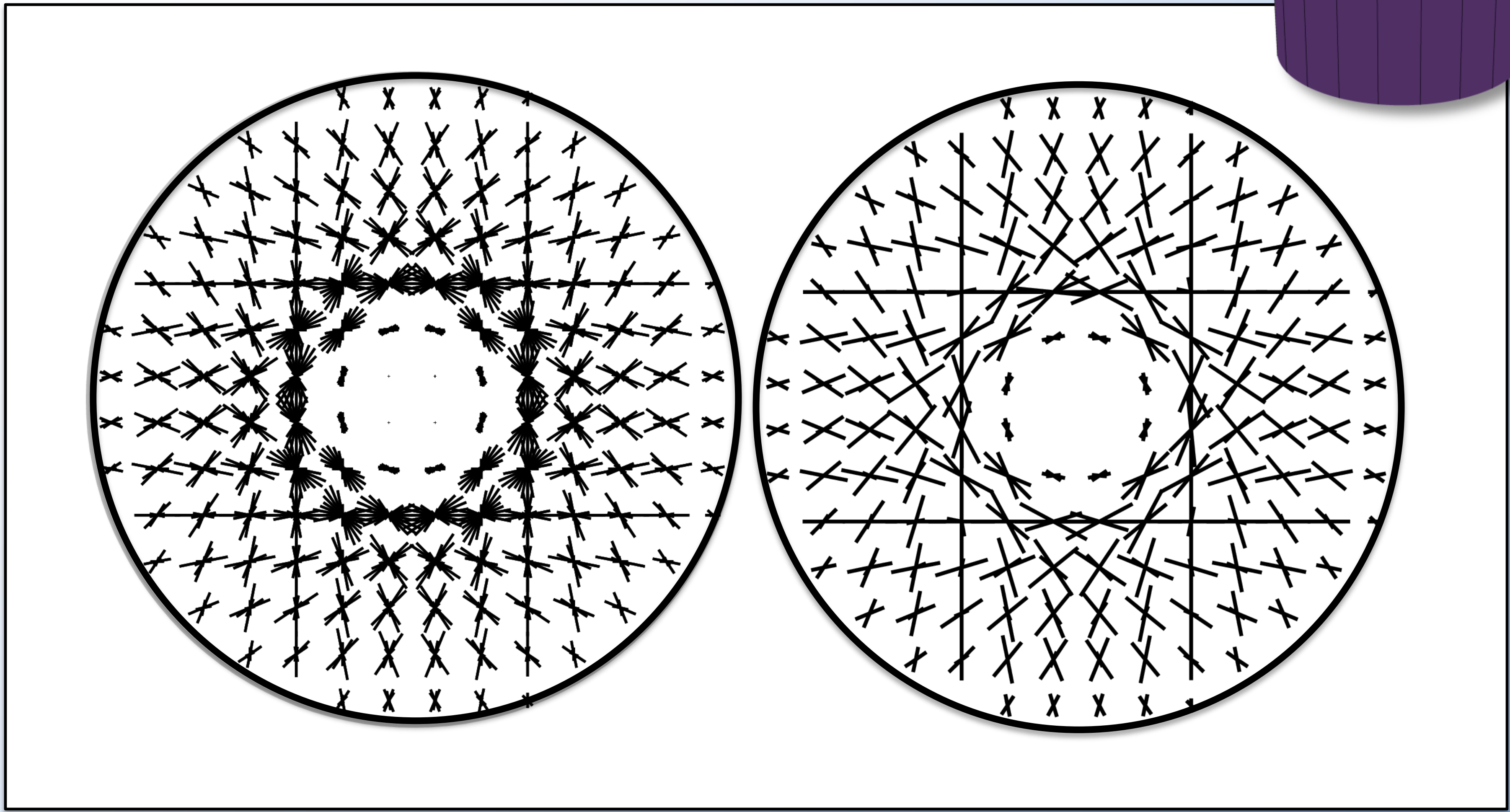


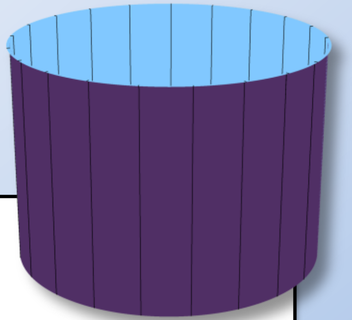


$_{C}(r) \;=\;$ ↵ $\cos(k_1 \cdot r) + \beta \cos(k_2 \cdot r)$

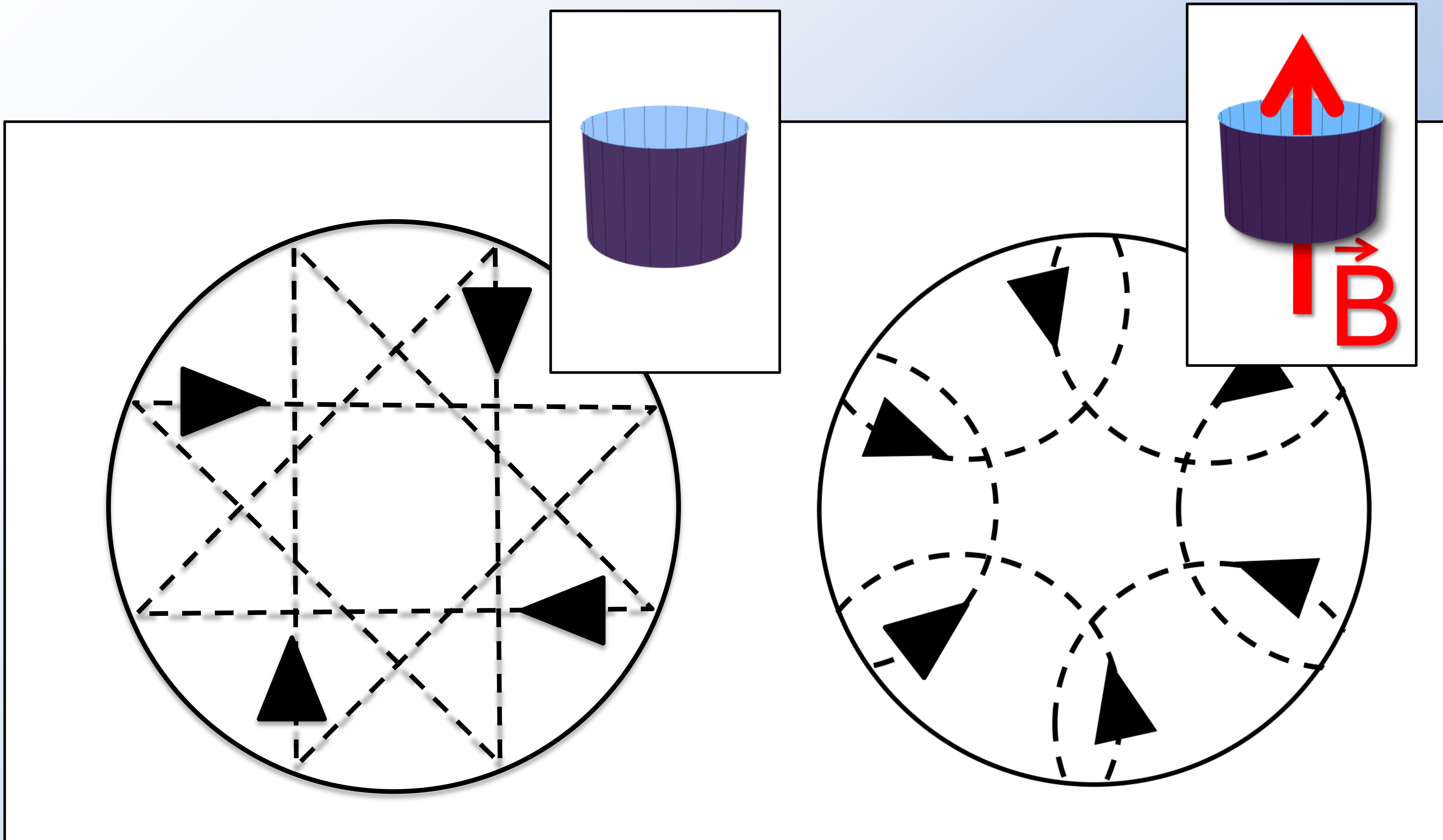


(a)
Ψ_C
(b)
(c)
(d)
(e)

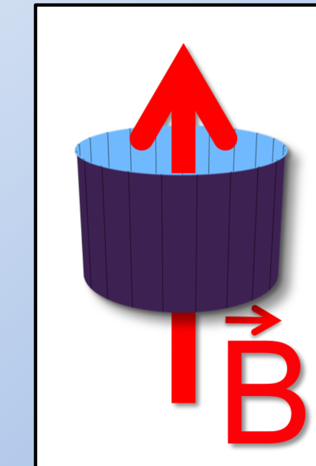




Magnetic Fields



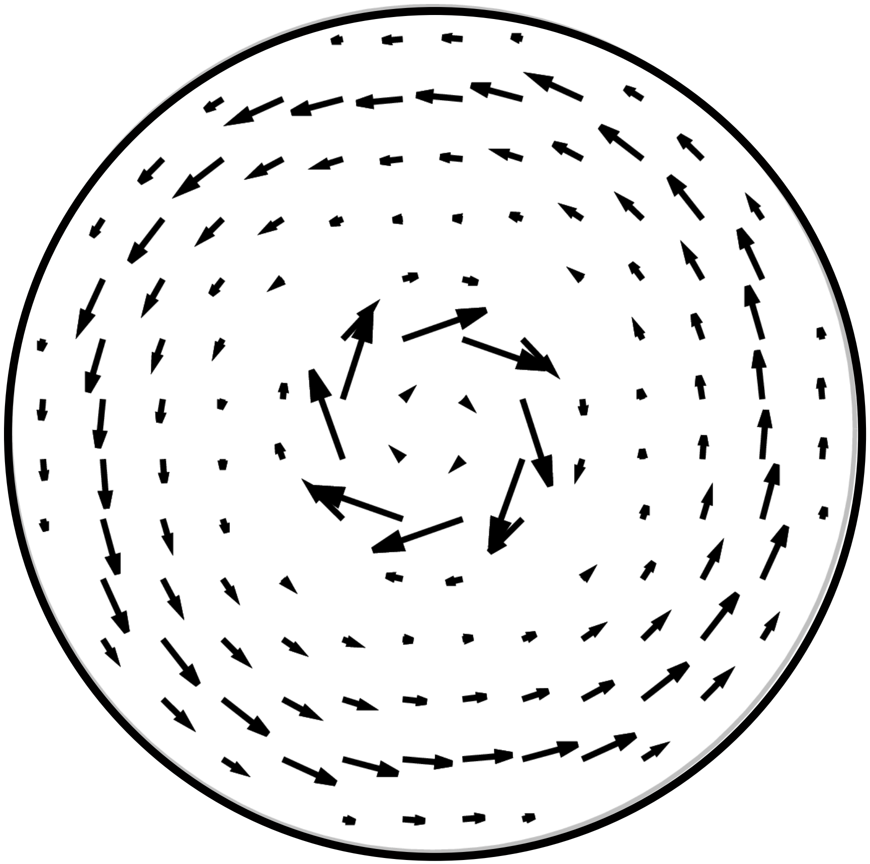
Magnetic Fields



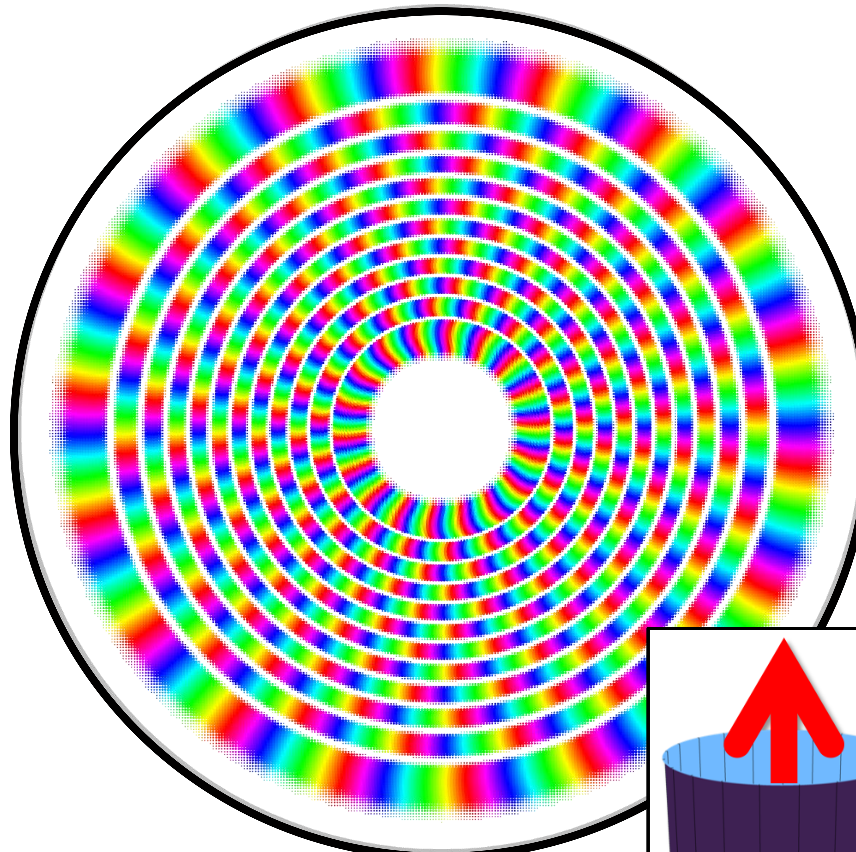
$$\mathrm{p} \; ! \quad \mathrm{p} - q\mathrm{A}/c$$

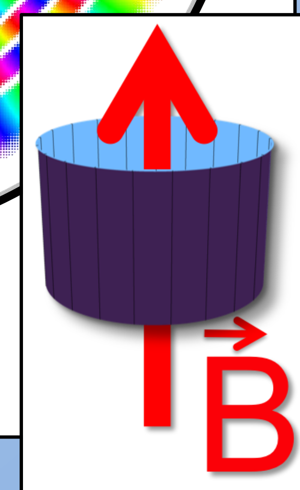
$$\mathrm{Hu}\,(\mathrm{r}_0, \mathrm{k}_0, \sigma; \quad (\mathrm{r})) \;=\; |h \;\; |\, \mathrm{r}_0, \mathrm{k}_0, \sigma i\,|^2$$

Flux



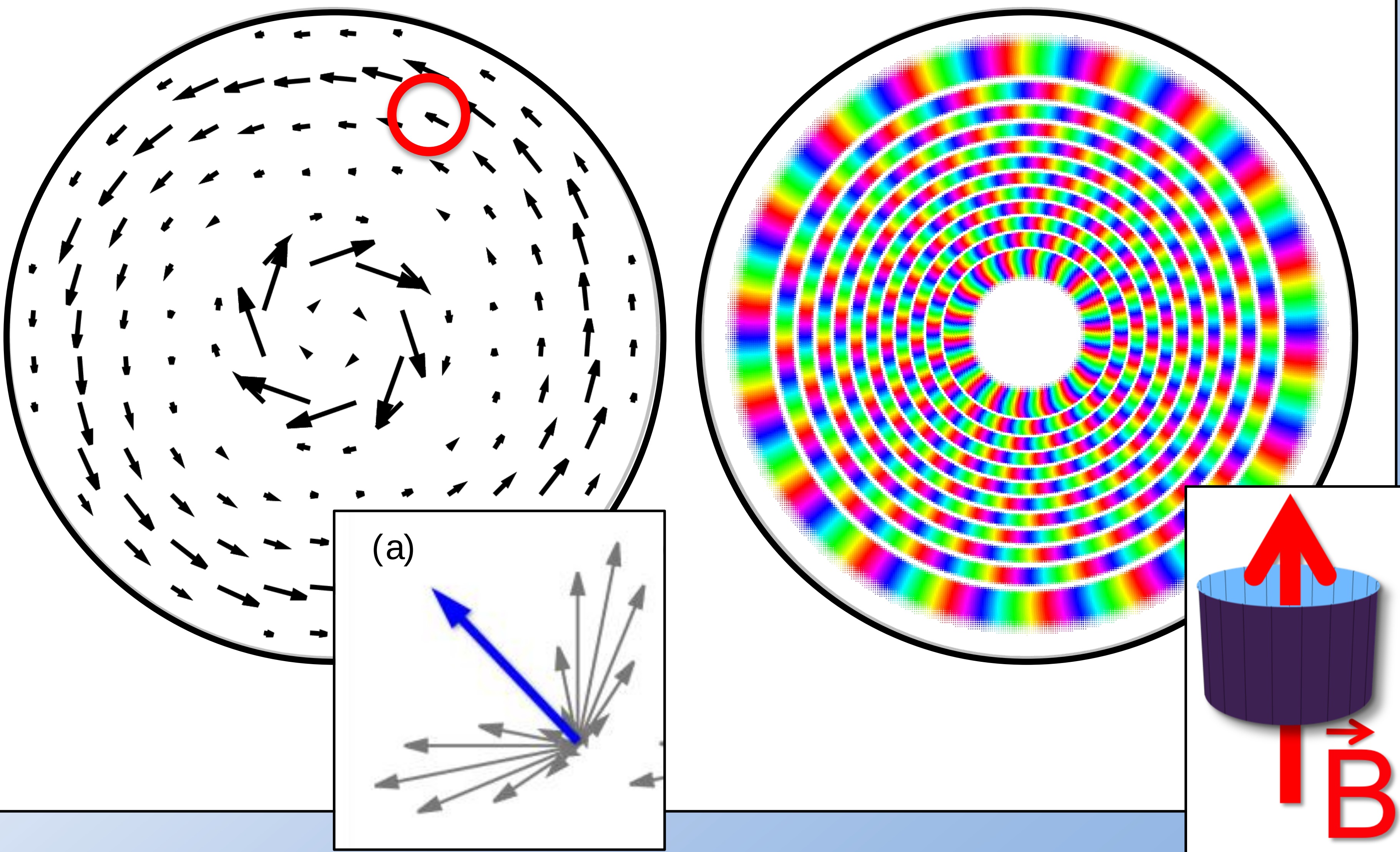
Wavefunction

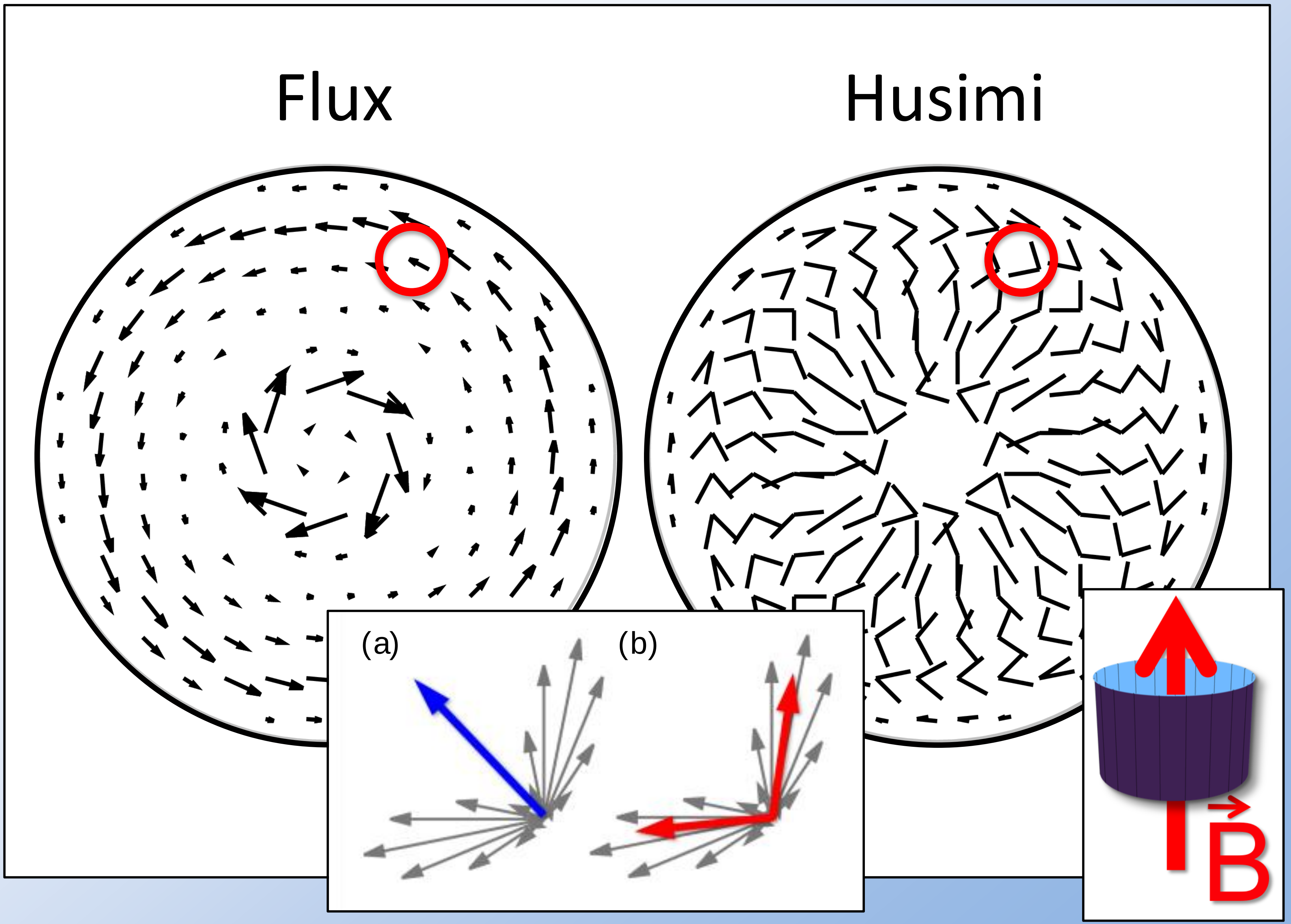


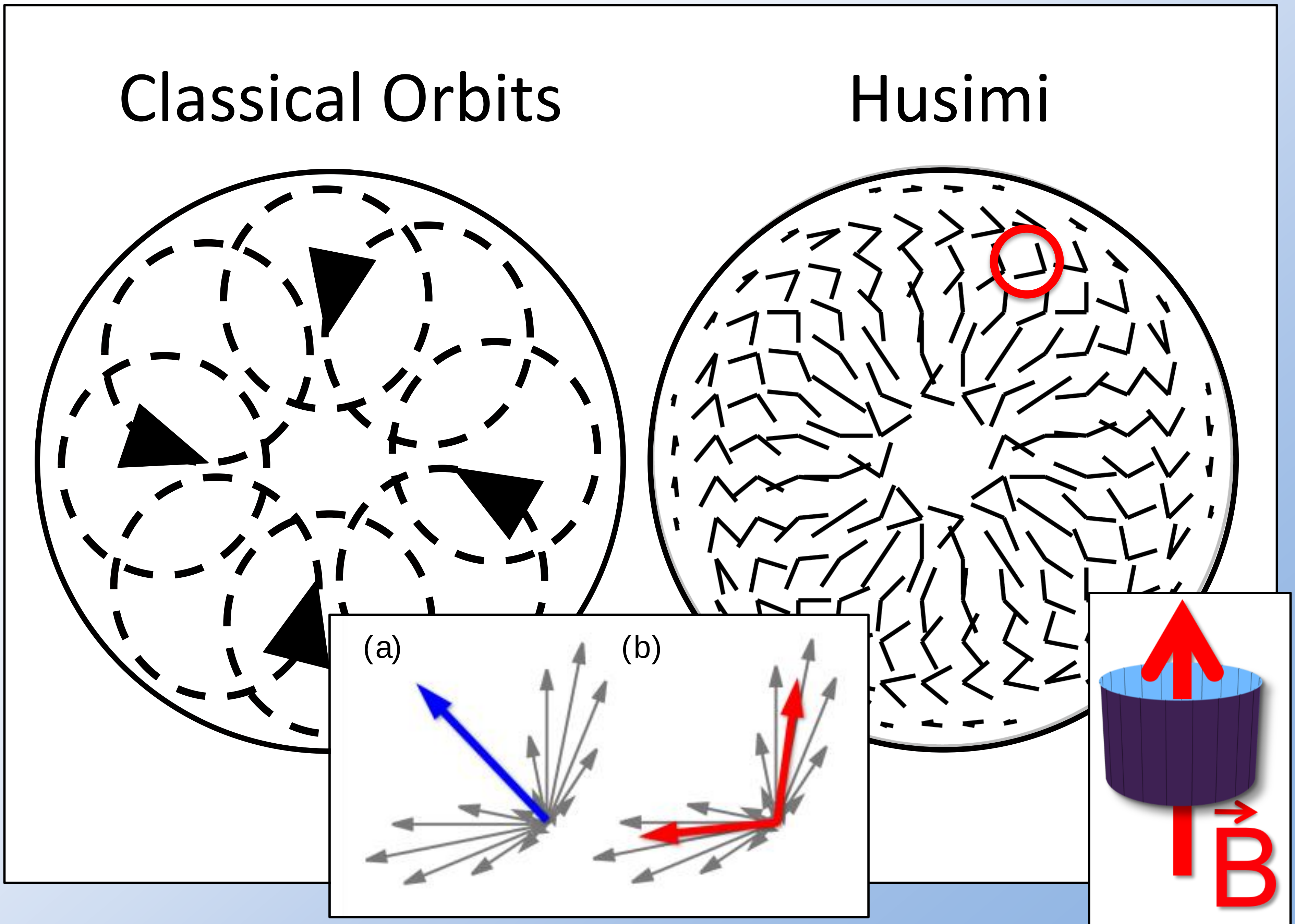


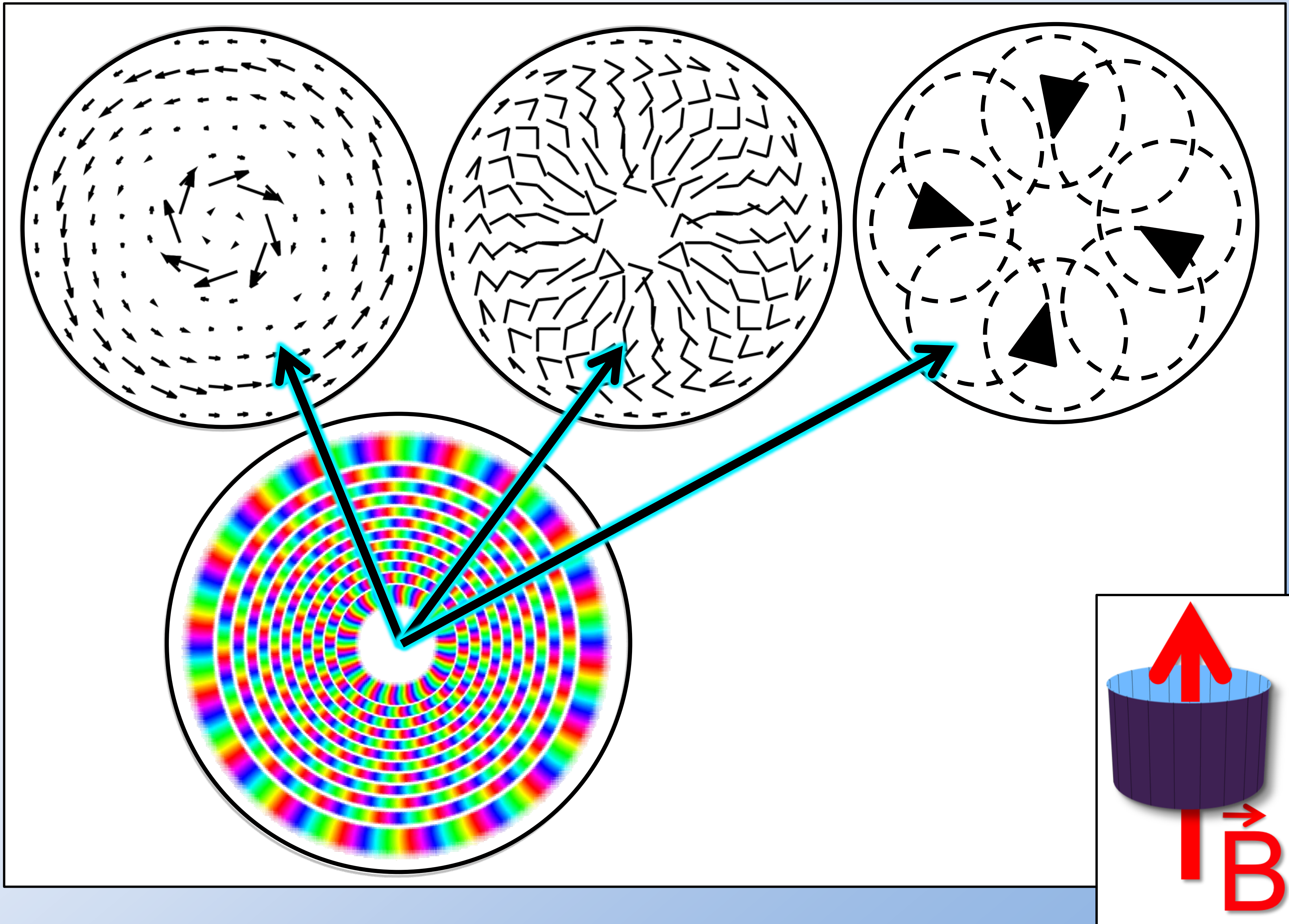
Flux

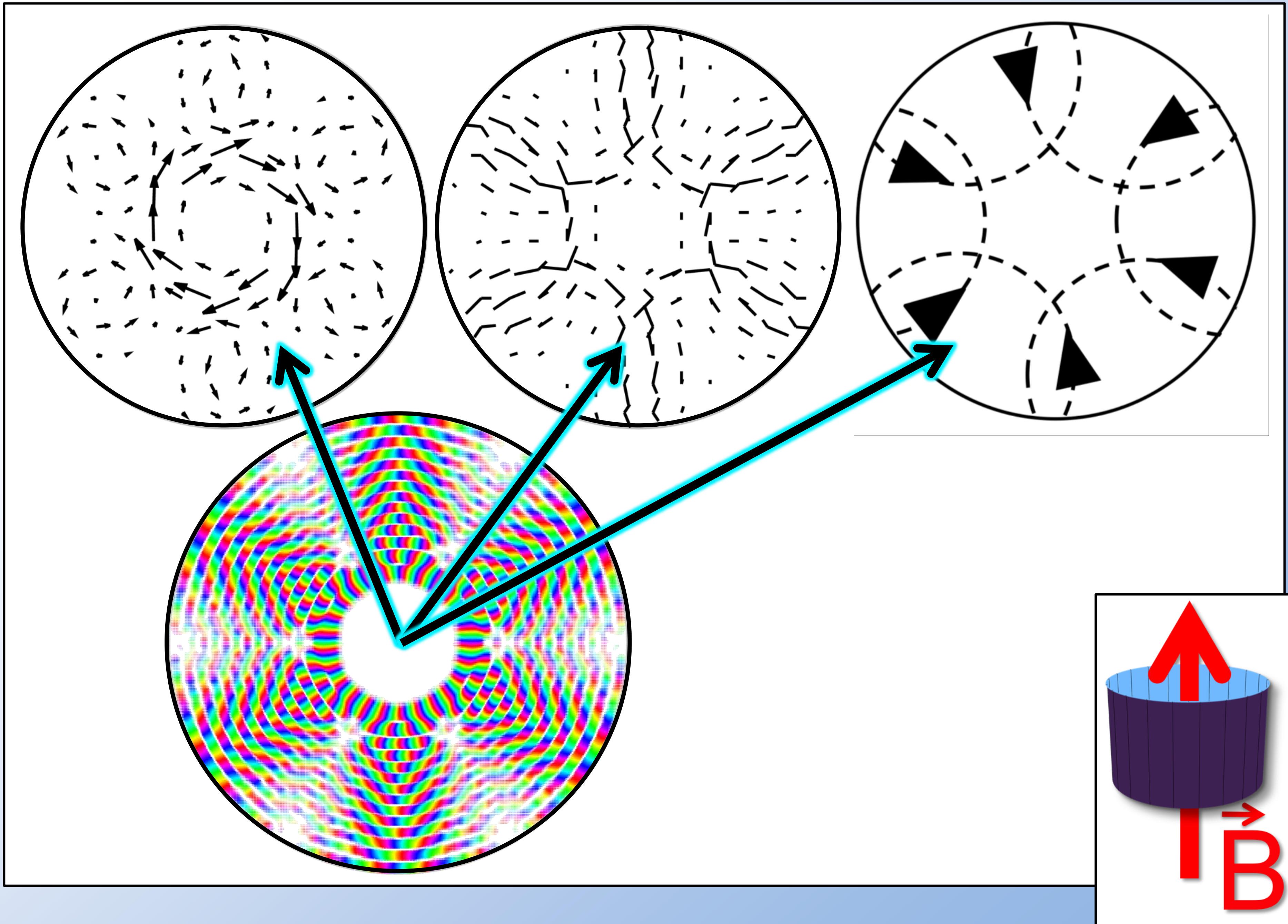
Wavefunction




Flux
Husimi
(a)
(b)
B


Classical Orbits
Husimi
(a)
(b)
B


$\vec{B}$


$\vec{B}$

More Examples at

www.douglasjmason.com

Acknowledgments

- http://www-heller.harvard.edu/people.html
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