Breaking Ground in Earthquake Simulation:

Extended Finite Element Methods for Repeated Earthquake Rupture

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Columbia University

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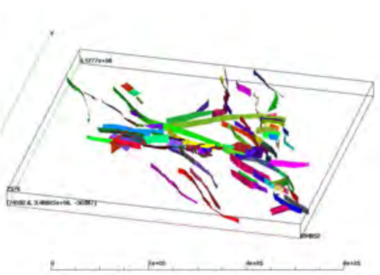
Introduction

San Andreas Fault Region



Community Fault Model
(Carl Gable, LANL)

Earthquakes

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \sigma(\mathbf{u})$$

- no interpenetration:

$$[\![\mathbf{u} \cdot \widehat{\mathbf{n}}]\!]_{\Gamma} = 0$$

- stick (Γ^D):

$$x \in \Gamma^N \; s.t. \left[\!\left[\frac{\partial \mathbf{u}}{\partial t}\right]\!\right]_{\Gamma^N} = 0 \Longrightarrow x \in \Gamma^D$$

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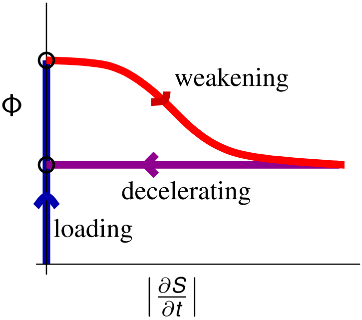

Solve earthquake dynamics on many faults with complicated, varying geometries and stick-slip friction boundary conditions.

- varying spatial scales ($10^{-1} - 10^{6} m$)
- varying temporal scales ($10^{0} - 10^{9} s$)
- highly heterogeneous material properties
- coupled physics (fluids, mantle)

Typical of hard, interesting problems in Earth Sciences!

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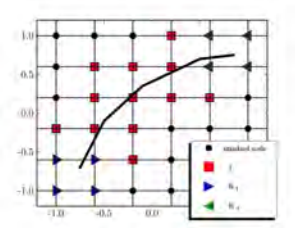
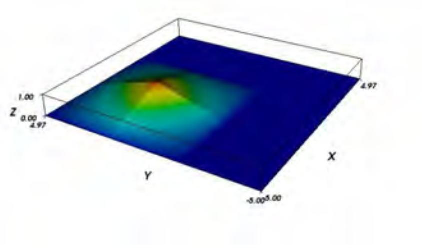
- Most methods require the generation of finite element meshes that conform to the faults.
- Generating meshes that conform to the geometry, have good aspect ratios, and adequately resolve regions with complex fault topology is an unsolved problem (in 3D).

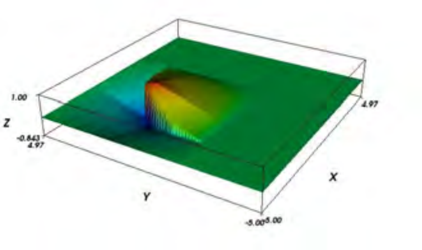
Meshless methods allow for discontinuities while not fitting a mesh to the geometry. Elements are extended with functions that take advantage of a priori *knowledge of solution characteristics.*



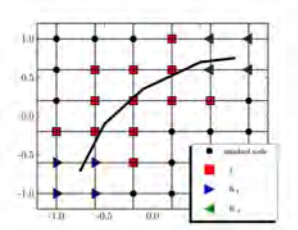
(Belytschko & Black, 1999)
(Dolbow, Moes, Belytschko 2000)

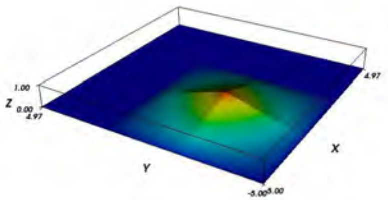


Standard Basis Function



Extended Basis Function





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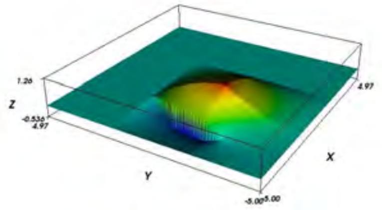
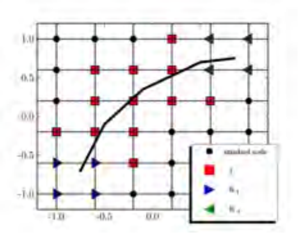
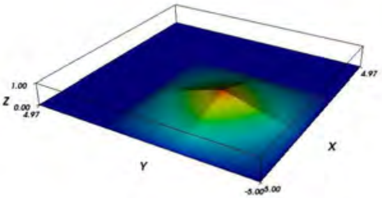


Tip-extended Basis Function

(Belytschko & Black, 1999)
(Dolbow, Moes, Belytschko 2000)





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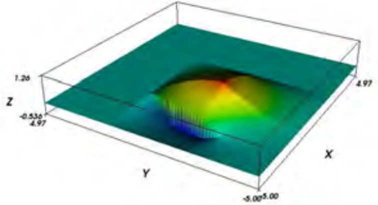

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(Belytschko & Black, 1999)
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BCs for meshless methods (Nitsche, 1971)

problem: Dirichlet-type BCs are difficult to apply, as no

$$x_j \in \Gamma \ s.t. \ \Psi_i(x_j) = \delta_{ij}$$

idea: Consider functional which incorporates boundary conditions:

$$\mathcal{J}[w] = \int_{\Omega} |\epsilon(w) : \sigma(w)|^2 - 2\int_{\Gamma} w \cdot \sigma(w) + \frac{\lambda\beta_\lambda}{h}\int_{\Gamma^D} w^2 + \frac{\mu\beta_\mu}{h}\int_{\Gamma^D} (w \cdot \widehat{\mathbf{n}})^2$$

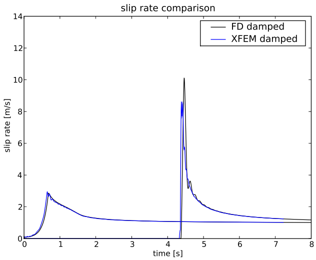
Minimizing the functional $\mathcal{J}[u^h - u]$ with true solution u over all $u^h \in \mathcal{V}^h \subset H^1$ is equivalent to the variational form (with our boundary conditions and the dynamic term):

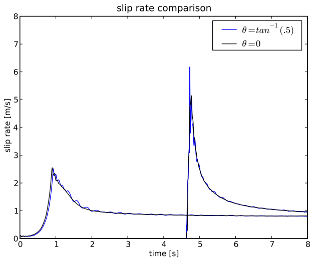
$$\int_{\Omega} v \cdot u^h_{tt} + \int_{\Omega} \epsilon(v) : \sigma(u^h) - \int_{\Gamma^D} u^h \cdot \sigma(v) + v \cdot \sigma(u^h)$$

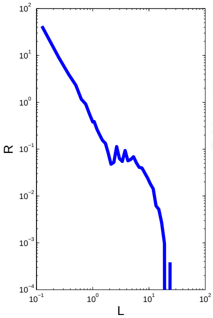
$$+ \frac{\lambda\beta_\lambda}{h}\int_{\Gamma^D} u^h_t \cdot v + \frac{\mu\beta_\mu}{h}\int_{\Gamma^D} (v \cdot \widehat{\mathbf{n}})(u^h_t \cdot \widehat{\mathbf{n}}) = \int_{\Gamma^N} v \cdot T$$

Coercive, symmetric operator $\longrightarrow$ symmetric, positive-definite matrix (though slightly ill-conditioned)

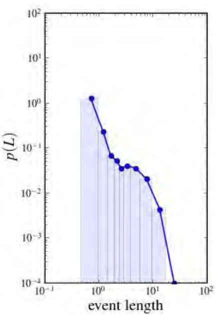
Validation and Verification





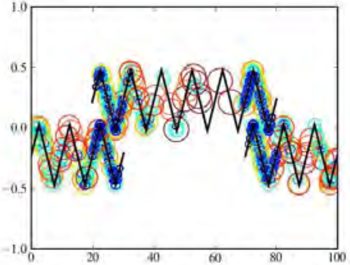


FD

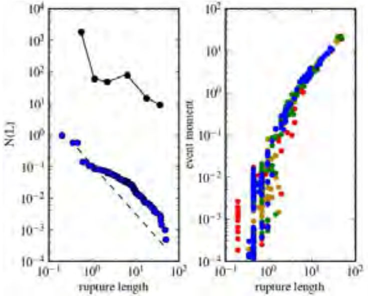


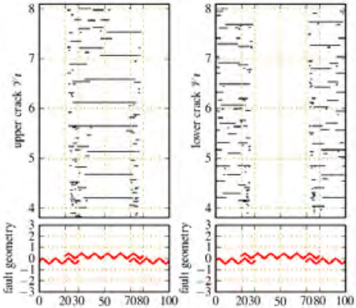
XFEM

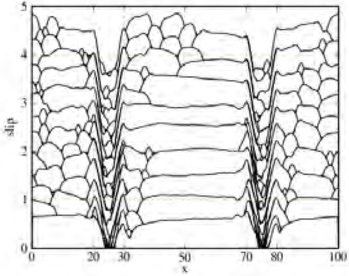
Results: two rough faults



seismicity







Soapbox

- Learn and practice good programming techniques, models, and tools (modular programming, good documentation, version control, etc.)
  - Others may want to use/read your code.
  - *You* may want to use/read your code.
  - You may want to add more physics, or take parts of your code for inclusion in other codes
- V & V: validation and verification (benchmark benchmark benchmark!)
- "All the hard work is in [linear solves]. The others [nonlinear solves, timestepping, physics control] are just cheap control loops." - David Keyes
- "Hybrid programming" – write computationally expensive code in lower-level, compiled languages (C, Fortran) (or better yet, use libraries) and write control code in higher-level, interpreted languages (Python).

Conclusions

- Geophysical problems are cool! They provide challenges for analysis, algorithms, implementation, data mining, and visualization.
- XFEM can provide an alternative to the meshing problem for simulations of repeated rupture in earthquake modeling.
- Time spent now on learning to program and developing good habits will result in savings in time and effort and better quality research later.

Future work must incorporate these techniques and basis functions into existing community earthquake code. Further complications may/will come from 3D.

Acknowledgements

- Bruce Shaw (LDEO)
- Marc Spiegelman (LDEO/Columbia)
- David Moulton (LANL)

DOE CSGF

Define Ω^i, an open cover on Ω, and ϕ_i, a partition of unity, such that:

$$\Omega \subset \bigcup_i \Omega^i \quad (1)$$

$$\text{support}(\phi_i) \subset \Omega^i \quad (2)$$

$$\sum_i \phi_i(x) = 1 \; \forall \, x \in \Omega \quad (3)$$

Choose a set of extension functions, $\mathcal{P}^i \subset H^1(\Omega^i)$ to define the PUFEM space:

$$\mathcal{P} \equiv \sum_i \mathcal{P}^i \phi_i = \left\{ \sum_i v_i \phi_i \mid v_i \in \mathcal{P}^i \right\} \subset H^1$$

Take both $\mathcal{U} \equiv \mathcal{V} \equiv \mathcal{P}$.