Modeling of Multiphase Flow in Porous Medium Systems

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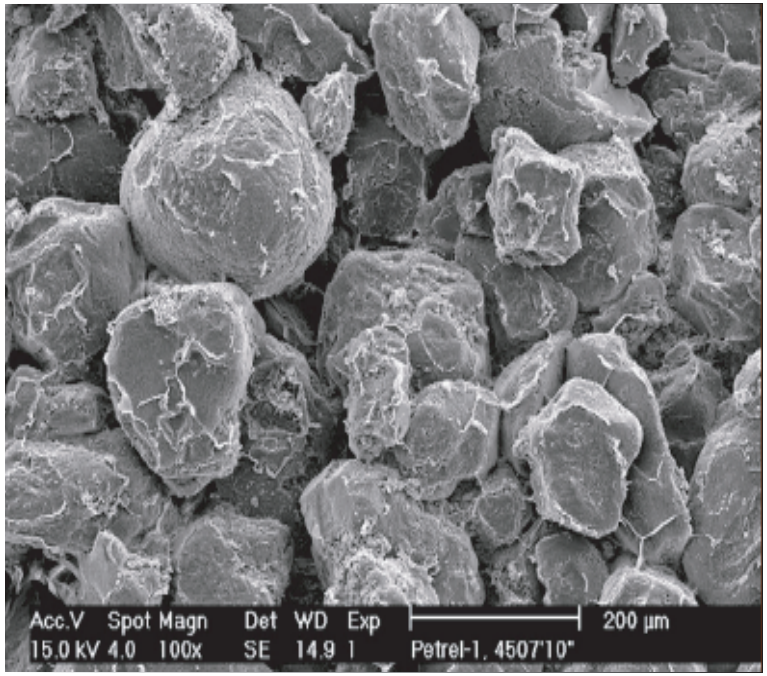
What is a porous medium system?

Porous Medium



- solid permeated by interconnected 3-D network of channels
- commonly referred to as pore spaces
- morphology and topology
- occur naturally in nature and industry

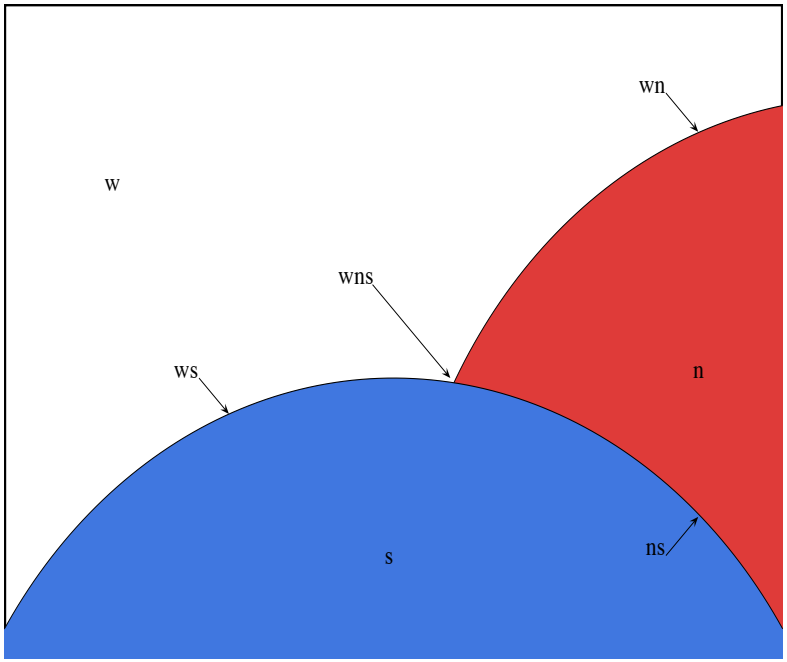
What is a multiphase porous medium system?

Multiphase system of interest - entities for a two fluid-phase system

phases : w, n, s

interfaces: wn, ws, ns

common curves: wns



Classic Solutions

- majority of porous medium flow simulators based on continuum theory
- consists of two main parts:
  1. conservation equations
  2. closure relations
- traditional approaches exist and are still widely used
- limits to effectiveness and predictability - motivation for new approaches

Traditional models:

$$\frac{\partial(\epsilon^\iota \rho^\iota)}{\partial t} = -\nabla\cdot(\epsilon^\iota \rho^\iota \mathbf{v}^\iota) + \mathcal{I}^\iota + \mathcal{S}^\iota$$

$$\mathbf{q}^\iota = \epsilon^\iota \mathbf{v}^\iota = \frac{-\kappa\kappa^{r\iota}}{\mu^\iota}\left(\nabla p^\iota - \rho^\iota \mathbf{g}\right)$$

$$\rho^\iota = \rho^\iota\left(p^\iota\right)$$

$$s^w = s^w\left(p^c\right)$$

- simplifying assumptions
- minimum set of conservation equations
- multiphase extension to Darcy's law
- define equations of state
- constitutive relations among pressure, saturation, and permeability

Thermodynamically Constrained Averaging Theory (TCAT)

- averages from smaller scale to scale of interest
- averages not only conservation equations but also thermodynamics and equilibrium conditions
- includes evolution equations for interfaces and common curves
- uses a constrained averaged entropy inequality and force-flux pair approach to guide closure relations

$$\sum_{\iota \in \mathcal{I}} \left(\mathcal{S}^{\iota} + \lambda^{\iota}_{\mathcal{M}} \mathcal{M}^{\iota} + \boldsymbol{\lambda}^{\iota}_{\boldsymbol{\mathcal{P}}} \cdot \boldsymbol{\mathcal{P}}^{\iota} + \lambda^{\iota}_{\mathcal{E}} \mathcal{E}^{\iota} + \lambda^{\iota}_{\mathcal{T}} \mathcal{T}^{\iota} \right) = \Lambda \geq 0$$

- clearly defines all variables, separates between exact forms and approximations, details assumptions
- creates a framework for extension, revision, and simplification

Two-Fluid-Phase Model

- 31 unknowns
- 2 conservation of mass eqns. and 18 momentum eqns.

$$\epsilon\frac{\partial s^{\iota}}{\partial t}+s^{\iota}\left\{\left[\hat{\alpha}^{\mathrm{b}}+(1-\epsilon)\hat{\beta}^{s}\right]\frac{\partial\langle\mathbf{n}_s\cdot\mathbf{t}_s\cdot\mathbf{n}_s\rangle_{\Omega_{ss},\Omega_{ss}}}{\partial t}+\epsilon\hat{\beta}^{\iota}\frac{\partial p^{\iota}}{\partial t}\right\}+\nabla\cdot(s^{\iota}\epsilon\mathbf{v}^{\bar{\iota},\bar{s}})=0$$

$$\text{and}\ \sum_{\kappa\in\mathcal{I}}\hat{\mathbf{R}}^{\iota}_{\kappa}\cdot\mathbf{v}^{\bar{\kappa},\bar{s}}=-\epsilon^{\iota}\rho^{\iota}\nabla\left(\psi^{\bar{\iota}}+\mu^{\bar{\iota}}\right)\qquad\text{for}\ \iota\in\{w,n\}$$

$$\hat{\mathbf{R}}^{\iota}_{\iota}\cdot\mathbf{v}^{\bar{\iota},\bar{s}}=\sum_{(\kappa\in\mathcal{I})\cap(\kappa\neq\iota)}\hat{\mathbf{R}}^{\iota}_{\kappa}\cdot\mathbf{v}^{\bar{\kappa},\bar{s}}\qquad\text{for}\ \iota\in\{wn,ws,ns,wns\}$$

Two-Fluid-Phase Model

- 3 constraints from the entropy inequality

$$-\frac{1}{\theta}\left\{x_s^{ws}\left[p_w^{ws}+\gamma^{ws}J_s^{\overline{\overline{ws}}}\right]+x_s^{ns}\left[p_n^{ns}+\gamma^{ns}J_s^{\overline{\overline{ns}}}\right]+\langle\mathbf{n}_s\cdot\mathbf{t}_s\cdot\mathbf{n}_s\rangle_{\Omega_{ss},\Omega_{ss}}\right.$$
$$\left.+x_s^{wns}\left[\gamma^{wns}\kappa_N^{\overline{\overline{wns}}}-\gamma_{wn}^{wns}\sin\Psi^w\right]\right\}=\hat{c}^s\frac{\mathrm{D}^{\bar{s}}\epsilon^s}{\mathrm{D}t} \tag{1}$$

$$p_w^{wn}-p_n^{wn}-\gamma^{wn}J_w^{\overline{\overline{wn}}}=\hat{c}^{wn}\left[\frac{\mathrm{D}^{\bar{s}}\epsilon^w}{\mathrm{D}t}+x_s^{ws}\frac{\mathrm{D}^{\bar{s}}\epsilon^s}{\mathrm{D}t}\right]$$
$$=\hat{c}^{wn}\left[\epsilon\frac{\mathrm{D}^{\bar{s}}s^w}{\mathrm{D}t}+\left(s^w-x_s^{ws}\right)\frac{\mathrm{D}^{\bar{s}}\epsilon}{\mathrm{D}t}\right] \tag{2}$$

and

$$-\frac{1}{\theta}\left[\gamma_{ws}^{wns}+\gamma_{wn}^{wns}\cos\Psi^w-\gamma_{ns}^{wns}+\gamma^{wns}\kappa_G^{\overline{\overline{wns}}}\right]=$$
$$\hat{c}^{wns}\left[\frac{\mathrm{D}^{\bar{s}}\epsilon^{ws}}{\mathrm{D}t}-J_s^{\overline{\overline{ws}}}x_s^{ws}\frac{\mathrm{D}^{\bar{s}}\epsilon^s}{\mathrm{D}t}\right] \tag{3}$$

Two-Fluid-Phase Model: Geometric Constraints

- 3 geometric contraints

$$\frac{\mathrm{D}^{\bar{s}}(\epsilon^{ws}+\epsilon^{ns})}{\mathrm{D}t} + J^{\bar{\bar{s}}}\frac{\mathrm{D}^{\bar{s}}\epsilon}{\mathrm{D}t} = -\nabla\cdot\left(\epsilon^{ws}\mathbf{G}^{ws}\right)\cdot\mathbf{v}^{\overline{ws},\bar{s}} - \epsilon^{ws}\mathbf{G}^{ws}\mathbf{:d}^{\overline{\overline{ws}}}$$
$$-\nabla\cdot\left(\epsilon^{ns}\mathbf{G}^{ns}\right)\cdot\mathbf{v}^{\overline{ns},\bar{s}} - \epsilon^{ns}\mathbf{G}^{ns}\mathbf{:d}^{\overline{\overline{ns}}} \tag{4}$$

$$\frac{\mathrm{D}^{\bar{s}}\epsilon^{wn}}{\mathrm{D}t} - \left(\epsilon^{ws}+\epsilon^{ns}\right)\cos\Psi^{w}\frac{\mathrm{D}^{\bar{s}}x_s^{ws}}{\mathrm{D}t} - J_w^{\overline{\overline{wn}}}\frac{\mathrm{D}^{\bar{s}}\epsilon^{w}}{\mathrm{D}t}$$
$$+\left[x_s^{ws}J_w^{\overline{\overline{wn}}} - x_s^{wns}\sin\Psi^{w}\right]\frac{\mathrm{D}^{\bar{s}}\epsilon}{\mathrm{D}t}$$
$$= x_s^{ns}\cos\Psi^{w}\left[\nabla\cdot\left(\epsilon^{ws}\mathbf{G}^{ws}\right)\cdot\mathbf{v}^{\overline{ws},\bar{s}} + \epsilon^{ws}\mathbf{G}^{ws}\mathbf{:d}^{\overline{\overline{ws}}}\right]$$
$$-x_s^{ws}\cos\Psi^{w}\left[\nabla\cdot\left(\epsilon^{ns}\mathbf{G}^{ns}\right)\cdot\mathbf{v}^{\overline{ns},\bar{s}} + \epsilon^{ns}\mathbf{G}^{ns}\mathbf{:d}^{\overline{\overline{ns}}}\right]$$
$$-\nabla\cdot\left(\epsilon^{wn}\mathbf{G}^{wn}\right)\cdot\mathbf{v}^{\overline{wn},\bar{s}} - \epsilon^{wn}\mathbf{G}^{wn}\mathbf{:d}^{\overline{\overline{wn}}} \tag{5}$$

$$
\begin{aligned}
\frac{\mathrm{D}^{\bar{s}}\epsilon^{wns}}{\mathrm{D}t} &+ \kappa_G^{\overline{\overline{wns}}}(\epsilon^{ws}+\epsilon^{ns})\frac{\mathrm{D}^{\bar{s}}x_s^{ws}}{\mathrm{D}t} - x_s^{wns}\kappa_N^{\overline{\overline{wns}}}\frac{\mathrm{D}^{\bar{s}}\epsilon}{\mathrm{D}t} \\
&= -\kappa_G^{\overline{\overline{wns}}}x_s^{ns}\nabla\cdot(\epsilon^{ws}\mathbf{G}^{ws})\cdot\mathbf{v}^{\overline{ws},\bar{s}} - \kappa_G^{\overline{\overline{wns}}}x_s^{ns}\epsilon^{ws}\mathbf{G}^{ws}\mathbf{:d}^{\overline{\overline{ws}}} \\
&\quad +\kappa_G^{\overline{\overline{wns}}}x_s^{ws}\nabla\cdot\left(\epsilon^{ns}\mathbf{G}^{ns}\right)\cdot\mathbf{v}^{\overline{ns},\bar{s}} + \kappa_G^{\overline{\overline{wns}}}x_s^{ws}\epsilon^{ns}\mathbf{G}^{ns}\mathbf{:d}^{\overline{\overline{ns}}} \\
&\quad -\nabla\cdot\left(\epsilon^{wns}\mathbf{G}^{wns}\right)\cdot\mathbf{v}^{\overline{wns},\bar{s}} - \epsilon^{wns}\mathbf{G}^{wns}\mathbf{:d}^{\overline{\overline{wns}}}
\end{aligned}
\tag{6}
$$

Two-Fluid-Phase Model: EOS

- 5 EOS

$$\hat{\alpha}^{\mathrm{b}} = -\frac{1}{(1-\epsilon)\rho^s}\left(\frac{\partial[(1-\epsilon)\rho^s]}{\partial\langle \mathbf{n}_s \cdot \mathbf{t}_s \cdot \mathbf{n}_s\rangle_{\Omega_{ss},\Omega_{ss}}}\right)_{\theta^{\overline{\overline{s}}}} \tag{7}$$

$$\gamma^{wn} J_w^{\overline{\overline{wn}}} = p^c(s^w, \epsilon^{wn}, x_s^{ws}) \tag{8}$$

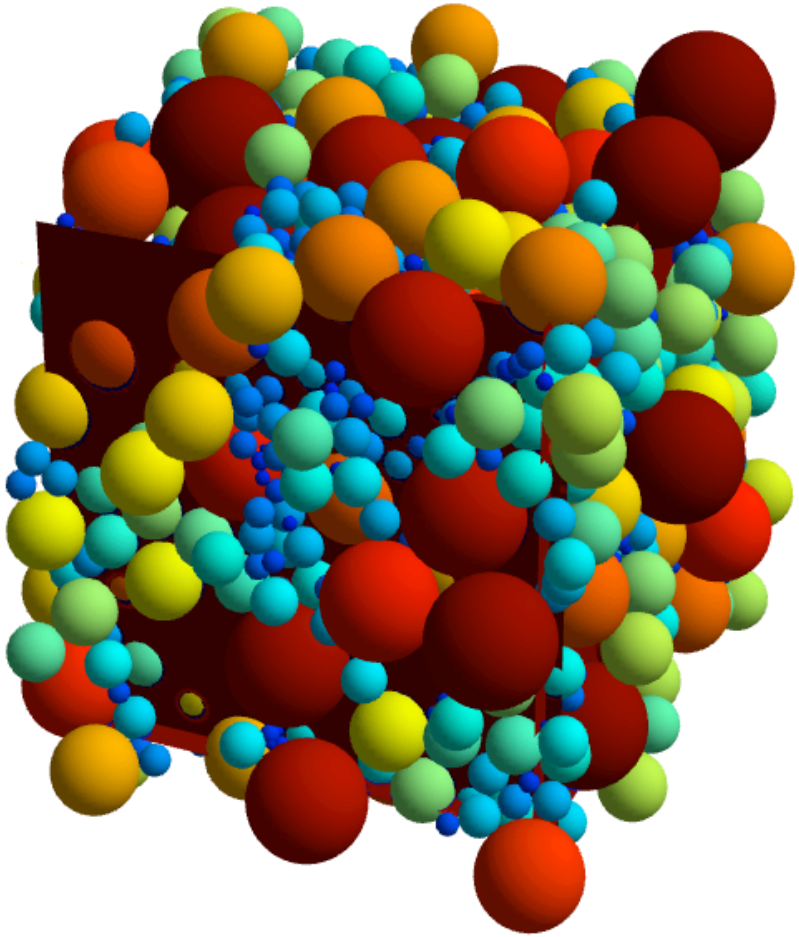
$$\kappa_N^{\overline{\overline{wns}}} = \kappa_N^{\overline{\overline{wns}}}(s^w, \epsilon^{ws}, \epsilon^{wns}) \tag{9}$$

$$\kappa_G^{\overline{\overline{wns}}} = \kappa_G^{\overline{\overline{wns}}}(s^w, x_s^{ws}, x_s^{wns}) \tag{10}$$

and

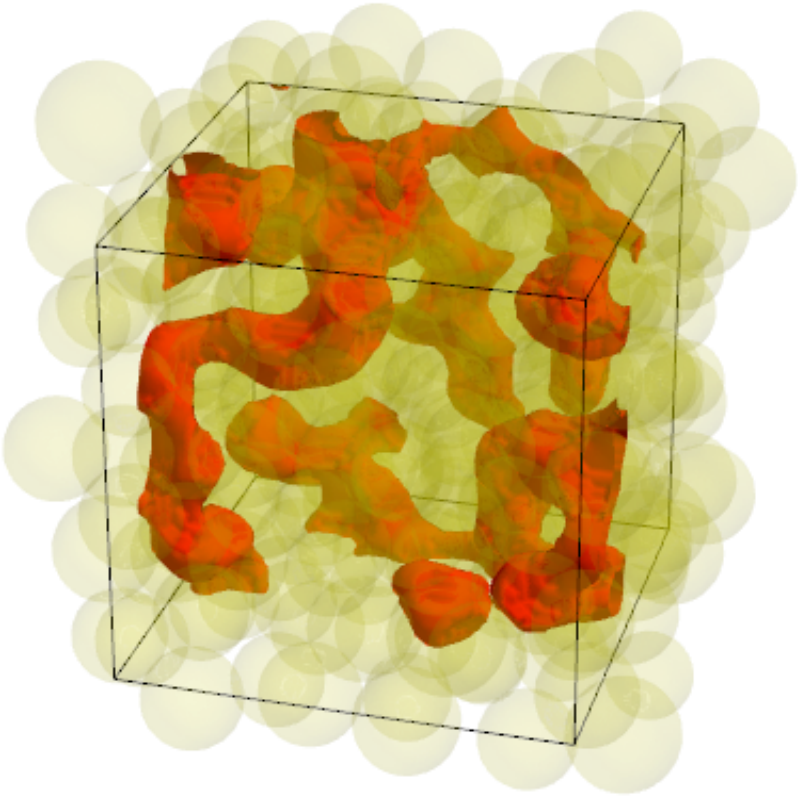
$$\Psi^w = \Psi^w(s^w, \epsilon^{wn}, x_s^{ws}) \tag{11}$$

Random Packing



- random packing algorithm
- non-overlapping, random-size, gravitationally stable spheres
- matches porosity, mean and standard deviations of log-normal grain diameter distributions

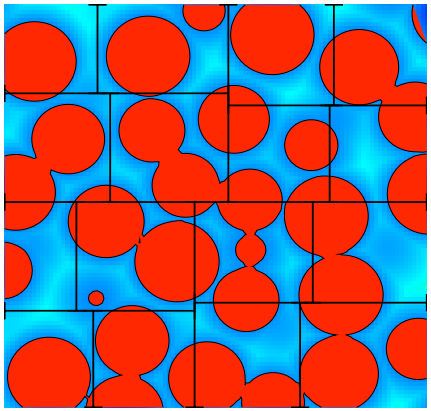
two-fluid phase system



- solves micrscopic Boltzmann equation
- D3Q19 lattice
- pressure boundary conditions
- multiphase formulation
  - external force at each lattice node as a function of neighboring node properties
  - long-range interactions between phases defined according to fluid-fluid or solid-fluid interaction

Simulations

- use pore-scale simulations to examine microscale physics and inform closure relations
- domain decomposition algorithm used to optimize work load for parallelization



- essentially linear parallel scaling
- NERSC
- currently working on determining the desired set of porous medium systems to model

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Averaging Operator

Averaging operator

$$\langle \mathcal{P}_\iota \rangle_{\Omega_j,\Omega_k,w} = \frac{\int\limits_{\Omega_j} w\mathcal{P}_\iota \, \mathrm{d}\mathfrak{r}}{\int\limits_{\Omega_k} w \, \mathrm{d}\mathfrak{r}} \tag{12}$$

intrinsic average: $\rho^\iota = \langle \rho_\iota \rangle_{\Omega_\iota,\Omega_\iota}$

mass average: $\mathbf{v}^{\bar{\iota}} = \langle \mathbf{v}_\iota \rangle_{\Omega_\iota,\Omega_\iota,\rho_\iota}$

specially defined averages: $\mathbf{t}^{\bar{\bar{\iota}}} = \left\langle \mathbf{t}_\iota - \rho_\iota \left(\mathbf{v}_\iota - \mathbf{v}^{\bar{\iota}}\right)\left(\mathbf{v}_\iota - \mathbf{v}^{\bar{\iota}}\right) \right\rangle_{\Omega_\iota,\Omega_\iota}$

Scales

- Molecular scale: 10^{-11} to 10^{-9} m - diffusion, reactions
- Pore-scale or microscale: order of a typical pore size ranging from 10^{-5} to 10^{-3} m - continuous fluids
- Lab scale, macroscale, or continuum scale: instruments and techniques for measuring characteristic of porous media exist in this range from 10^{-2} to 10^{0} m - governing equations apply
- Field scale or meso-scale: order of an aquifer thickness ranging from 10^{1} to 10^{3} m - more practical for hydrological community
- Regional scale or mega-scale: regional water source management scale ranging from 10^{3} to 10^{5} or larger

Representative Elementary Volume (REV)

Representative Elementary Volume (REV)

An REV can be described as a region of porous media large enough to include all phases present and of a sufficient size such that the values of averages that characterize a phase are independent of that size.

The REV is considered to have a characteristic length scale that is much smaller than the system length scale such that gradients of macroscale quantities within the system are meaningful.

Step 1 - Generation of Entropy Inequality

General entropy balance equation:

$$\frac{\mathrm{D}^{\bar{\iota}}\eta^{\bar{\bar{\iota}}}}{\mathrm{D}t} + \eta^{\bar{\bar{\iota}}}\mathbf{I}\!:\!\mathbf{d}^{\bar{\bar{\iota}}} - \nabla\cdot\left(\epsilon^{\iota}\boldsymbol{\varphi}^{\bar{\bar{\iota}}}\right) - \epsilon^{\iota}b^{\iota} - \sum_{\kappa\in\mathcal{J}_{c\iota}}\left(\overset{\kappa\to\iota}{M_{\eta}} + \overset{\kappa\to\iota}{\Phi}\right) = \Lambda^{\bar{\bar{\iota}}} \quad (13)$$

Summing over all entities gives us:

$$\sum_{\iota\in\mathcal{J}}\mathcal{S}^{\iota} = \sum_{\iota\in\mathcal{J}}\left(\frac{\mathrm{D}^{\bar{\iota}}\eta^{\bar{\bar{\iota}}}}{\mathrm{D}t} + \eta^{\bar{\bar{\iota}}}\mathbf{I}\!:\!\mathbf{d}^{\bar{\bar{\iota}}} - \nabla\cdot\left(\epsilon^{\iota}\boldsymbol{\varphi}^{\bar{\bar{\iota}}}\right) - \epsilon^{\iota}b^{\iota}\right) = \Lambda \geq 0 \quad (14)$$

Step 2 - Conservation Equations

Mass:

$$\mathcal{M}^{\iota} = \frac{\mathrm{D}^{\bar{\iota}}\left(\epsilon^{\iota}\rho^{\iota}\right)}{\mathrm{D}t} + \epsilon^{\iota}\rho^{\iota}\mathbf{I}\!:\!\mathbf{d}^{\bar{\bar{\iota}}} - \sum_{\kappa\in\mathcal{I}_{c\iota}} \overset{\kappa\longrightarrow\iota}{M} = 0 \tag{15}$$

Momentum:

$$\boldsymbol{\mathcal{P}}^{\iota} = \frac{\mathrm{D}^{\bar{\iota}}\left(\epsilon^{\iota}\rho^{\iota}\mathbf{v}^{\bar{\iota}}\right)}{\mathrm{D}t} + \epsilon^{\iota}\rho^{\iota}\mathbf{v}^{\bar{\iota}}\mathbf{I}\!:\!\mathbf{d}^{\bar{\bar{\iota}}} - \nabla\cdot\left(\epsilon^{\iota}\mathbf{t}^{\bar{\bar{\iota}}}\right) - \epsilon^{\iota}\rho^{\iota}\mathbf{g}^{\bar{\iota}}$$

$$- \sum_{\kappa\in\mathcal{I}_{c\iota}} \left(\overset{\kappa\longrightarrow\iota}{\mathbf{M}_{v}} + \overset{\kappa\longrightarrow\iota}{\mathbf{T}}\right) = 0 \tag{16}$$

Step 2 - Conservation Equations (continued)

Energy:

$$\mathcal{E}^{\iota} = \frac{\mathrm{D}^{\bar{\iota}}\left[E^{\bar{\bar{\iota}}} + \epsilon^{\iota}\rho^{\iota}\left(\frac{1}{2}\mathbf{v}^{\bar{\iota}}\cdot\mathbf{v}^{\bar{\iota}} + K_E^{\bar{\bar{\iota}}} + \psi^{\bar{\iota}}\right)\right]}{\mathrm{D}t}$$

$$+\left[E^{\bar{\bar{\iota}}} + \epsilon^{\iota}\rho^{\iota}\left(\frac{1}{2}\mathbf{v}^{\bar{\iota}}\cdot\mathbf{v}^{\bar{\iota}} + K_E^{\bar{\bar{\iota}}} + \psi^{\bar{\iota}}\right)\right]\mathbf{I}\!:\!\mathbf{d}^{\bar{\bar{\iota}}} - \nabla\cdot\left(\epsilon^{\iota}\mathbf{t}^{\bar{\bar{\iota}}}\!\cdot\!\mathbf{v}^{\bar{\iota}} + \epsilon^{\iota}\mathbf{q}^{\bar{\bar{\iota}}}\right)$$

$$-\epsilon^{\iota}h^{\iota} - \sum_{\kappa\in\mathcal{J}_{c\iota}}\left(\overset{\kappa\to\iota}{M_E} + \overset{\kappa\to\iota}{T_v} + \overset{\kappa\to\iota}{Q}\right) = 0 \qquad (17)$$

35 conservation equations containing 171 unknowns

Step 3 - Averaging of Thermodynamics

- Classic Irreversible Thermodynamics
  CIT: $E(\mathbf{x}, t) = E[\eta(\mathbf{x}, t), \rho(\mathbf{x}, t)]$

- Thermodynamic equations for phases, interfaces and common curves

Fluid phases:

$$\begin{aligned}\mathcal{T}^{\iota} &= \frac{\mathrm{D}^{\bar{\iota}} E^{\bar{\bar{\iota}}}}{\mathrm{D}t} - \theta^{\bar{\bar{\iota}}} \frac{\mathrm{D}^{\bar{\iota}} \eta^{\bar{\bar{\iota}}}}{\mathrm{D}t} - \mu^{\bar{\iota}} \frac{\mathrm{D}^{\bar{\iota}} \left(\epsilon^{\iota} \rho^{\iota}\right)}{\mathrm{D}t} + p^{\iota} \frac{\mathrm{D}^{\bar{\iota}} \epsilon^{\iota}}{\mathrm{D}t} \\ &+ \left\langle \eta_{\iota} \frac{\mathrm{D}^{\bar{\iota}} \left(\theta_{\iota} - \theta^{\bar{\bar{\iota}}}\right)}{\mathrm{D}t} + \rho_{\iota} \frac{\mathrm{D}^{\bar{\iota}} \left(\mu_{\iota} - \mu^{\bar{\iota}}\right)}{\mathrm{D}t} - \frac{\mathrm{D}^{\bar{\iota}} \left(p_{\iota} - p^{\iota}\right)}{\mathrm{D}t} \right\rangle_{\Omega_{\iota}, \Omega} = 0\end{aligned} \tag{18}$$

Step 5 - Augmented Entropy Inequality

Augmented entropy inequality:

$$\sum_{\iota \in \mathcal{J}} \left(\mathcal{S}^{\iota} + \lambda^{\iota}_{\mathcal{M}} \mathcal{M}^{\iota} + \boldsymbol{\lambda}^{\iota}_{\boldsymbol{\mathcal{P}}} \cdot \boldsymbol{\mathcal{P}}^{\iota} + \lambda^{\iota}_{\mathcal{E}} \mathcal{E}^{\iota} + \lambda^{\iota}_{\mathcal{T}} \mathcal{T}^{\iota} \right) = \Lambda \geq 0, \qquad (19)$$

each term added to $\sum_{\iota \in \mathcal{J}} \mathcal{S}^{\iota}$ is equal to zero

Step 6 - Constrained Entropy Inequality

$$\sum_{\iota}\left\{\frac{\mathrm{D}^{\bar{\iota}}\eta^{\bar{\bar{\iota}}}}{\mathrm{D}t}+\lambda^{\iota}_{\mathcal{M}}\frac{\mathrm{D}^{\bar{\iota}}\left(\epsilon^{\iota}\rho^{\iota}\right)}{\mathrm{D}t}+\boldsymbol{\lambda}^{\iota}_{\boldsymbol{\mathcal{P}}}\cdot\frac{\mathrm{D}^{\bar{\iota}}\left(\epsilon^{\iota}\rho^{\iota}\mathbf{v}^{\bar{\iota}}\right)}{\mathrm{D}t}+\lambda^{\iota}_{\mathcal{E}}\left[\frac{\mathrm{D}^{\bar{\iota}}E^{\bar{\bar{\iota}}}}{\mathrm{D}t}+\mathbf{v}^{\bar{\iota}}\cdot\frac{\mathrm{D}^{\bar{\iota}}\left(\epsilon^{\iota}\rho^{\iota}\mathbf{v}^{\bar{\iota}}\right)}{\mathrm{D}t}\right.\right.$$

$$\left.+\left(K^{\bar{\bar{\iota}}}_{E}-\frac{\mathbf{v}^{\bar{\iota}}\cdot\mathbf{v}^{\bar{\iota}}}{2}+\psi^{\bar{\iota}}\right)\frac{\mathrm{D}^{\bar{\iota}}\left(\epsilon^{\iota}\rho^{\iota}\right)}{\mathrm{D}t}+\epsilon^{\iota}\rho^{\iota}\frac{\mathrm{D}^{\bar{\iota}}\left(K^{\bar{\bar{\iota}}}_{E}+\psi^{\bar{\iota}}\right)}{\mathrm{D}t}\right]+\lambda^{\iota}_{\mathcal{T}}\left[\frac{\mathrm{D}^{\bar{\iota}}E^{\bar{\bar{\iota}}}}{\mathrm{D}t}-\theta^{\bar{\bar{\iota}}}\frac{\mathrm{D}^{\bar{\iota}}\eta^{\bar{\bar{\iota}}}}{\mathrm{D}t}\right.$$

$$-\mu^{\bar{\iota}}\frac{\mathrm{D}^{\bar{\iota}}\left(\epsilon^{\iota}\rho^{\iota}\right)}{\mathrm{D}t}+p^{w}\frac{\mathrm{D}^{\overline{w}}\epsilon^{w}}{\mathrm{D}t}+p^{n}\frac{\mathrm{D}^{\overline{n}}\epsilon^{n}}{\mathrm{D}t}-\gamma_{ws}\frac{\mathrm{D}^{\overline{ws}}\epsilon^{ws}}{\mathrm{D}t}-\gamma_{wn}\frac{\mathrm{D}^{\overline{wn}}\epsilon^{wn}}{\mathrm{D}t}$$

$$\left.\left.-\gamma_{ns}\frac{\mathrm{D}^{\overline{ns}}\epsilon^{ns}}{\mathrm{D}t}+\gamma_{wns}\frac{\mathrm{D}^{\overline{wns}}\epsilon^{wns}}{\mathrm{D}t}\right]+\cdots\right\}=\Lambda\geq 0$$

Step 7 - Simplified Entropy Inequality

- assume independence of solid surface orientation
- assume macroscopically simple system
- assume slow solid-phase deformation
- assume independence for product-splitting integral approximations

Step 8 - Closure Relations

- resultant constitutive forms

$$\mathbf{t}^{\overline{\overline{w}}} + p^{w}\mathbf{I} = 0 \tag{20}$$

$$\mathbf{t}^{\overline{\overline{s}}} - \mathbf{t}^{s} = 0 \tag{21}$$

$$\mathbf{t}^{\overline{\overline{ws}}} - \gamma^{ws}\left(\mathbf{I} - \mathbf{G}^{ws}\right) = 0 \tag{22}$$

- linear approximation of flux - creates dependence of variables on system properties

LB Model

The evolution of the fluid particle distributions is governed by the discrete Boltzmann equation,

$$f_i(\mathbf{x}+\mathbf{e}_i, t+1) - f_i(\mathbf{x}, t) = \mathbf{S}\left[f_i^{(eq)}(\mathbf{x}, t) - f_i(\mathbf{x}, t)\right] + F_i \quad (23)$$

where f_i are the particle distribution functions associated with each $\mathbf{e}_i$ at each node $\mathbf{x}$, f_i^{eq} are the equilibrium distribution functions, $\mathbf{S}$ is a collision operator, and F_i is an external forcing term .
To simulate multiphase flow in porous media, long-range interactions of the form

$$\mathbf{F}_k = \mathbf{F}_{k,f-f} + \mathbf{F}_{k,f-s} + \rho_k \mathbf{g}_k \quad (24)$$

are included, where $\mathbf{F}_{k,f-f}$ is the fluid-fluid interaction force, $\mathbf{F}_{k,f-s}$ is the fluid-solid interaction force, and $\rho_k \mathbf{g}_k$ is the gravitational force for fluid k.