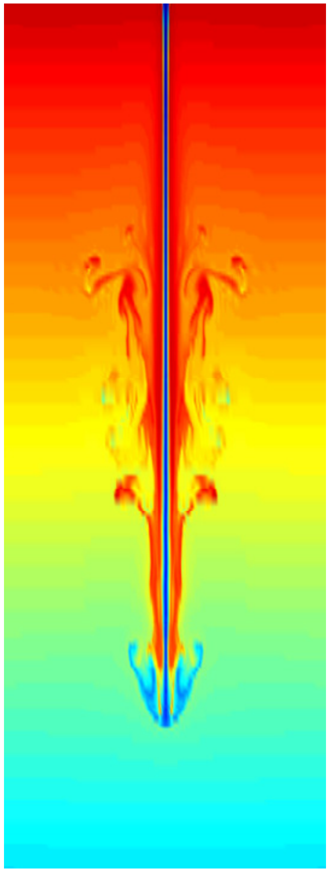
Blips, Coils, and Pedals:
Instabilities of buoyant jets in fluids with stable density stratification



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Abstract

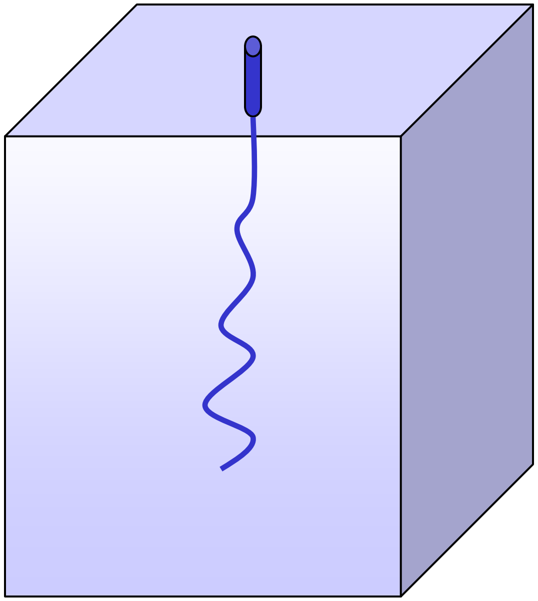
In biological flow experiments, algae gather in a negatively buoyant plume that breaks apart into round blips as it descends. The question of whether the motion of the individual cells is essential to the formation of the instability was posed. Recent three-dimensional and Hele-Shaw experiments have demonstrated that similar behaviors are found in dense jets descending into stably stratified fluids. These experiments have shown a rich array of dynamics including buckling and overturning, blip formation, shear type instabilities in the entrained conduit and multiple jet interactions. However, the flows are quite complex requiring viscous, buoyancy, convection and stratification effects to be considered together. Direct numerical simulation of the Boussinesq equations with high order spectral element methods allows the detailed aspects of the flows to be captured and understood without resorting to excessive approximations. We show that these behaviors are not tied to cell motions in biological flows, surface tensions effects, or Hele-Shaw cells and can be understood in terms of the parameter regimes being considered. Furthermore, we find a vortex street behavior that may be the pedal breakdown observed in other liquid into liquid jet experiments.

Overview

We consider the behavior of a dense jet injected into a fluid with a stable density stratification

Example application areas:

- Atmospheric circulation
- Ocean models
- Effluent discharges
- Smokestacks
- Bacterial Swarms



Historical Developments

- A.J. Reynolds, JFM Feb 15, 1962 "Observations of a liquid into liquid jet"
  - Observed (a) shearing puffs, (b) symmetric condensations, (c) sinuous undulations, (d) pedal breakdown and (e) confused breakdown depending on the injection rate.
  - Breakdown was observed a Reynolds number around 300 $Re = \frac{Vd}{\nu} = \frac{4Q}{\pi \nu d}$

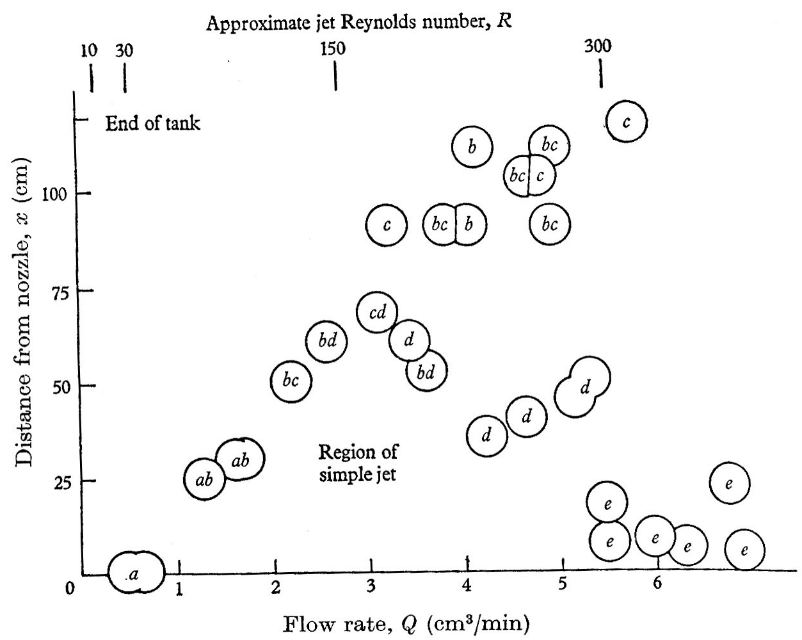


FIGURE 1. Positions of breakdown of simple jet plotted against flow rate, Q. In table 1 the symbols a, b, c, and d are related to modes of breakdown.

FIGURE 2. Form of breakdown near the nozzle at low flow rates: the shearing puffs, mode (a) of table 1.

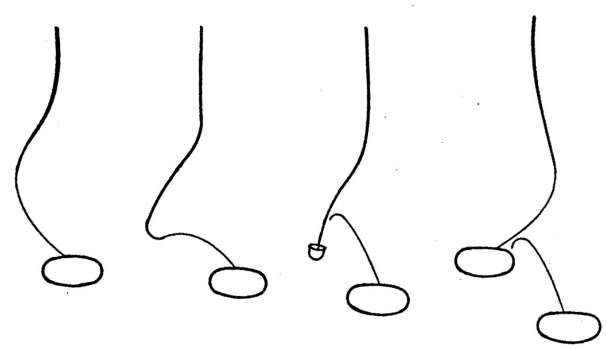
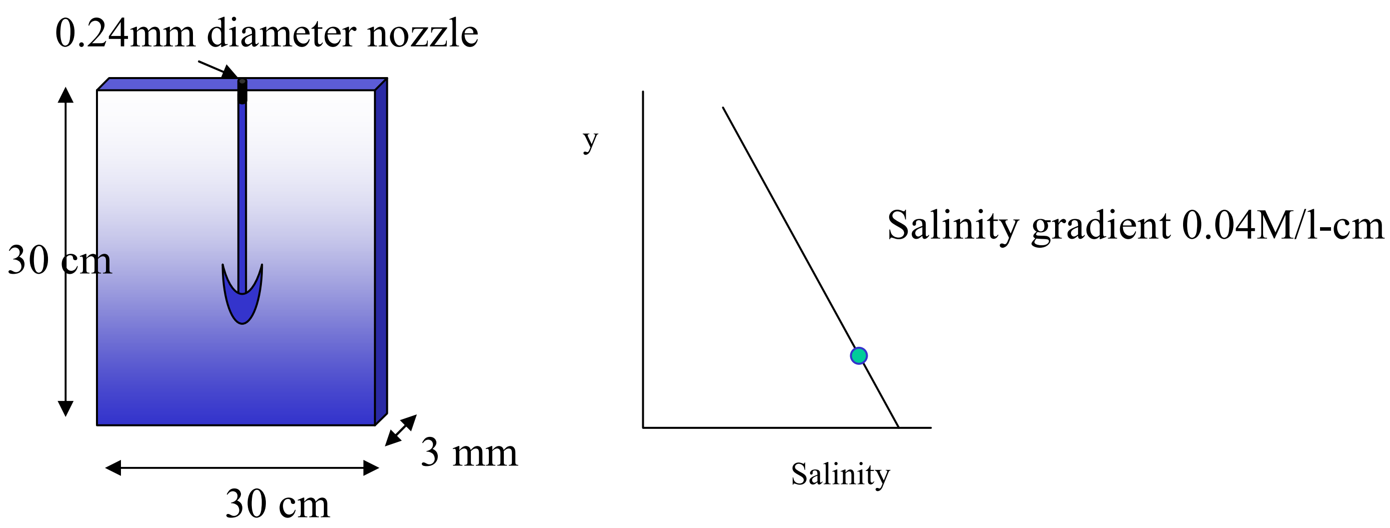


FIGURE 3. Successive stages of the pedal breakdown, mode (d) of table 1.

Hele-Shaw Experiments

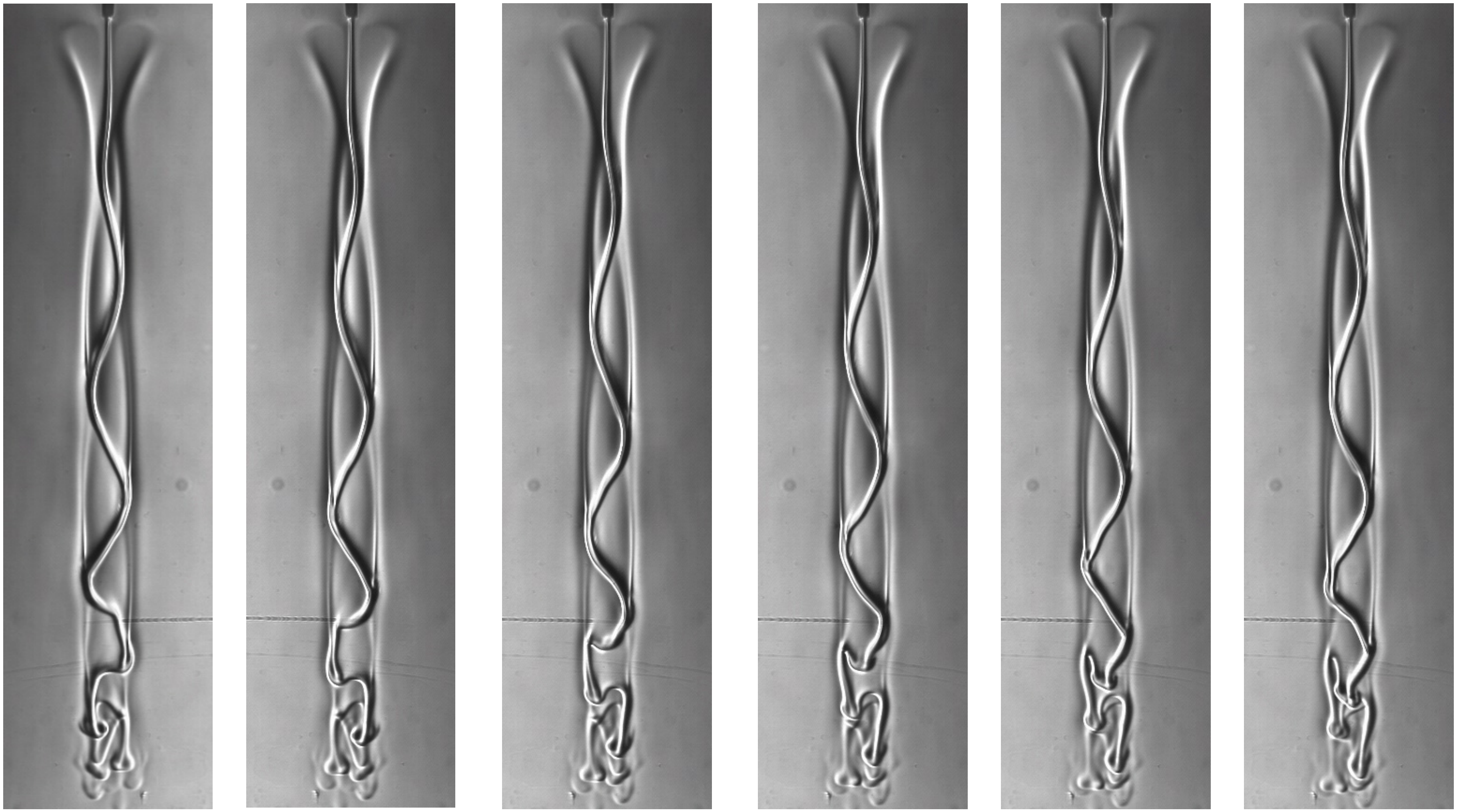
by A.L. Pesci, et al. PRL 2003

- A dense jet is injected into a tank with a stable density stratification.
- The jet is injected at the top from a needle with .24mm diameter at velocities of 0.02cm/s to 1 cm/s. (Re = 0.03 to 3)
- Visualized using the difference in index of refraction from concentration variations

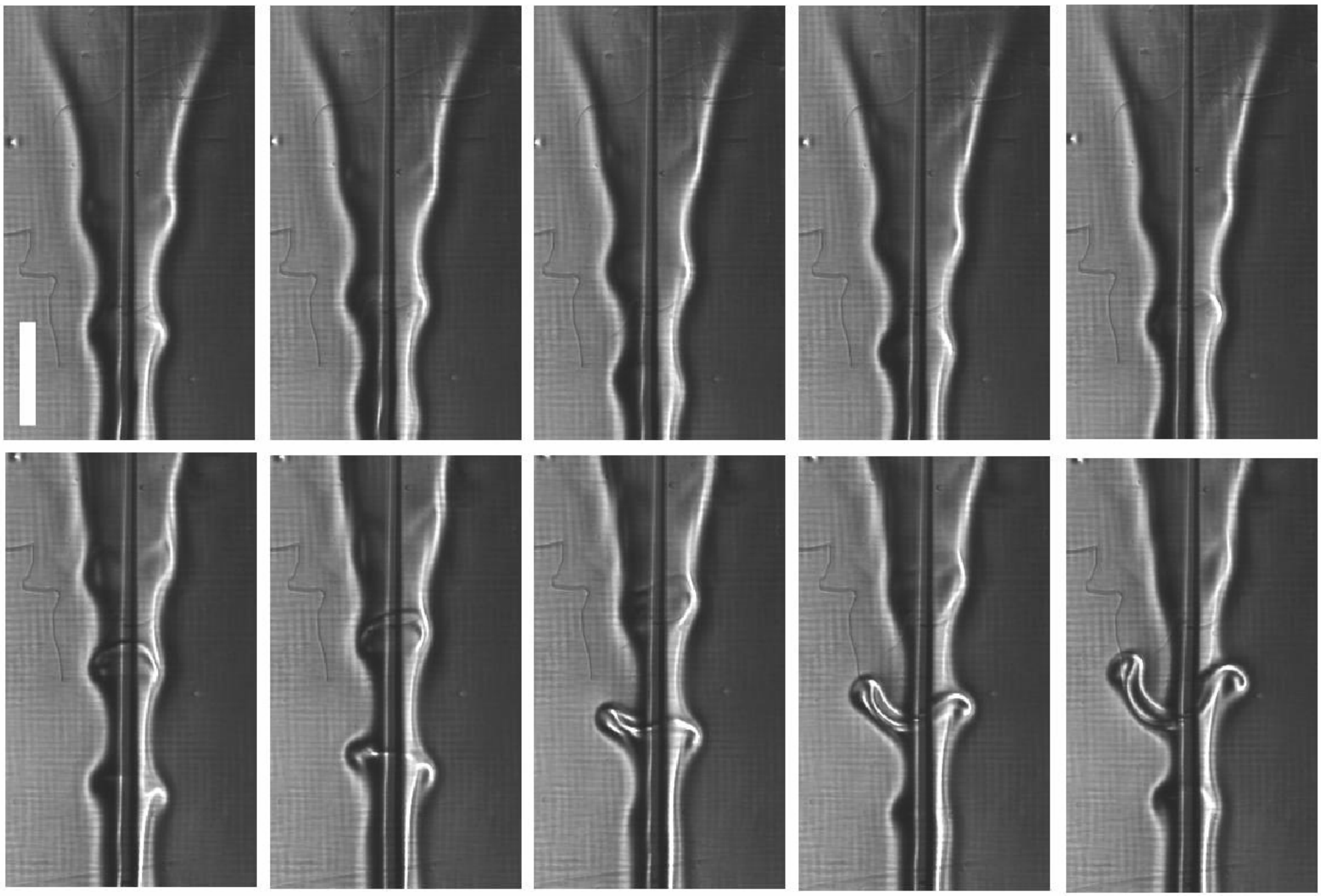


Hele-Shaw Experiments - Buckling

Jet.mov.lnk



Hele-Shaw Experiments – Conduit



Mathematical Formulation

•The small but nonzero Reynolds number does not permit the use of boundary layer approximations. T
•There is not a similarity structure to the governing equations making analysis difficult.
•Viscous, convection, and buoyancy effects must all be considered together.

Reynolds number $Re = \frac{U_0 L}{\nu}$

Grasshof number $Gr = \frac{g\beta\left(S_j - S_\infty(0)\right)L^3}{\nu^2}$

Schmidt number $Sc = \frac{\nu}{D}$

Mathematical Formulation and Direct Numerical Simulation (DNS)

The Boussinesq approximation to the incompressible Navier-Stokes equations was used:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + g\beta (S - S_\infty) + \nu \Delta \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

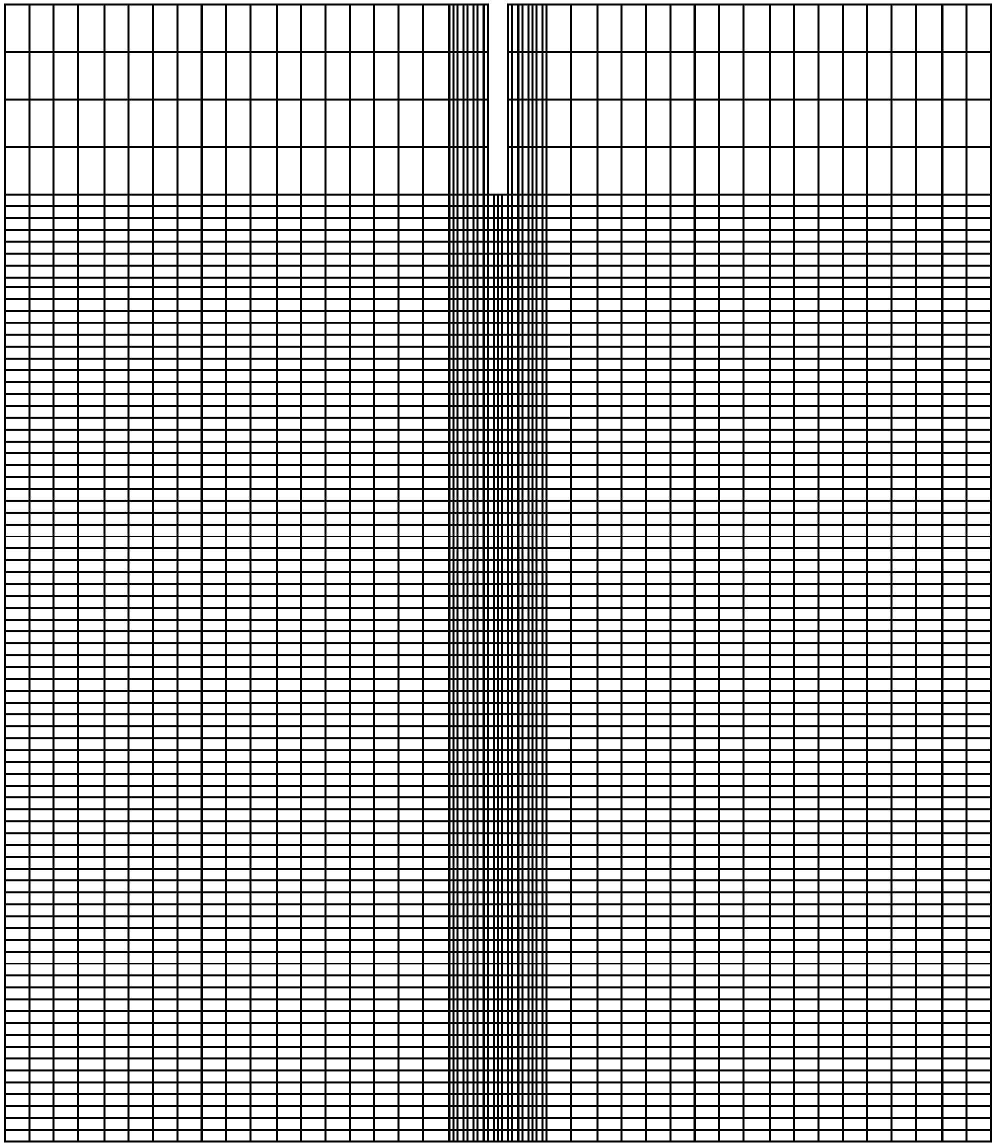
$$\frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S = D \Delta S$$

where **u** is velocity, S is salinity, κ is diffusivity, ν is kinematic viscosity, ρ is the density.

DNS was done with a high-order spectral element method. 4740 elements were used with ~330,000 grid points.

The relevant parameters are the injection rate, the density of injected jet, the density stratification, and the viscosity.

Direct Numerical Simulation



Spectral Element Method

Example (Maday and Patera, 1989)

1D Helmholtz problem

$$-(p(x)u')' + \lambda^2 u = f(x)$$

$$u(1) = u(-1) = 0$$

$$\tau_\infty > p(x) > \tau_0 > 0$$

$$x \in \Omega = [-1,1]$$

Ref:

Maday, Y. and Patera, A.T. “Spectral element methods for the Navier-Stokes equations,” in *State of the Art Surveys in Computational Mechanics*, A.K. Noor, ed., ASME, New York, pp. 71-143 (1989)

Deville, M.O, Fischer, P.F, and Mund E.H. High order methods for incompressible fluid flow. Cambridge University Press, 2002

Weak formulation

Look for a solution in the Sobolev Space

$$H_0^1 = \left\{ v \in L^2(\Omega), v' \in L^2(\Omega), v(\partial\Omega) = 0 \right\}$$

Recast the equation as a variational problem

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega p(x)u'(x)v'(x)dx + \lambda^2 \int_\Omega u(x)v(x)dx = \int_\Omega f(x)v(x)dx$$

$$\forall v \in H_0^1(\Omega), \quad a(u,v) = (f,v)$$

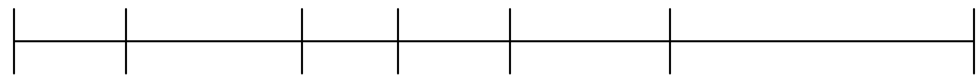
$$a(u,v) = \int_\Omega p(x)u'(x)v'(x)dx + \lambda^2 \int_\Omega u(x)v(x)dx$$

The domain can be mapped back to a reference domain (i.e. [-1,1]). This is usually done at the discrete level using an affine transformation or using the Gordon-Hall algorithm (for more complex deformation in higher dimensions)

Discretization

Split Ω into subintervals (spectral elements)

$$h = (N, K)$$



$$\Omega = \bigcup_{k=1}^{K} \Omega_k$$

$$P_N(\Omega_k) = \{\text{Polynomials of degree} \le N \text{ on } \Omega_k\}$$

Now look for a solution in this subspace X_h

$$\forall v \in X_h, \int_\Omega p(x) u_h'(x) v_h'(x)\,dx + \lambda^2 \int_\Omega u_h(x) v_h(x)\,dx = \int_\Omega f(x) v_h(x)\,dx$$

Gauss-Lobatto Quadrature

Approximate the inner product (u,v) with a finite sum.

Select the quadrature points as the Gauss-Lobatto or Gauss-Lobatto-Legendre points to integrate exactly polynomials of degree 2N-1.

$$\forall \phi \in P_{2N-1}(\Omega), \int_{\Omega} \phi(r)dr = \sum_{i=0}^{N} \rho_i \phi(\xi_i)$$

$$\xi_0 = -1, \xi_N = 1, \forall i = 1, \cdots, N-1; L'_N(\xi_i) = 0$$

$$u, v \in P_N([-1,1]), \int_{-1}^{1} u(r)v(r)dr = \sum_{i=0}^{N} \rho_i u(\xi_i)v(\xi_i)$$

Basis for Solution Space

Choose a basis for X_h, such as Lagrangian Interpolants where

$$h_i(\xi_i) = \delta_{ij}$$

Express functions in X_h in terms of basis functions

$$W_h^k(r) = \sum_{i=0}^{N} w_i^k h_i(r)$$

Enforce continuity $W_N^k = W_0^{k+1}, k = 1, \ldots, N-1$

Boundary conditions $W_0^1 = W_N^K = 0$

Note the choice of Gauss-Lobatto Lagrangian interpolants result in coupling only at the endpoints and a diagonal mass matrix.

$$\int_{-1}^{1} h_m(x) h_n(x) dx = \sum_{i=0}^{N} \rho_i h_m(\xi_i) h_n(\xi_i)$$

Tensor Product Matrices

For higher dimensions, use tensor product of GLL points

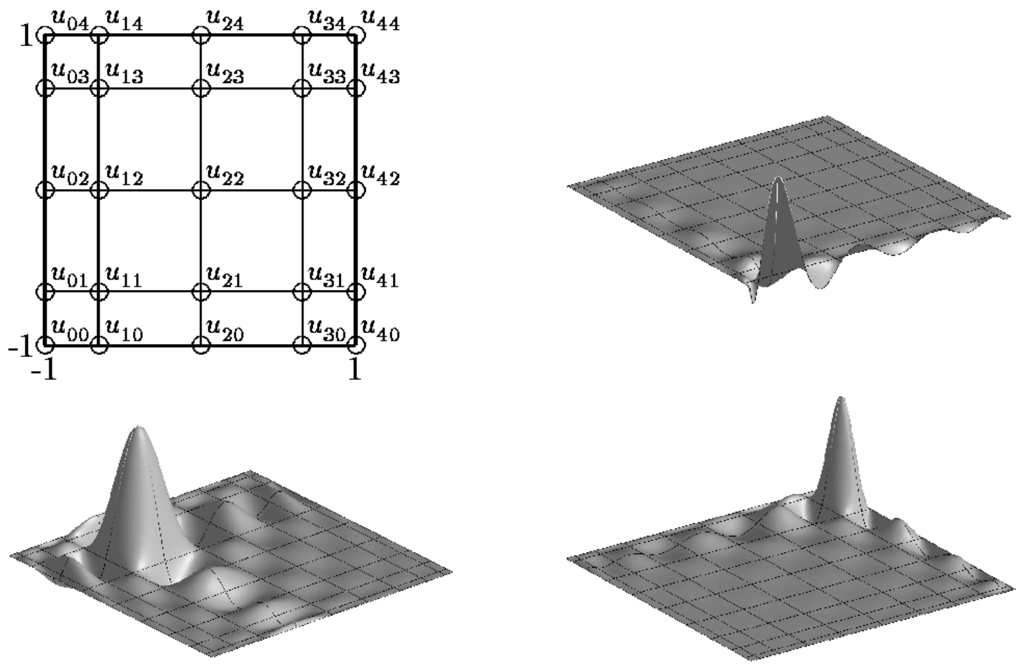


Figure 4.2.1: Clockwise from upper left: GLL nodal point distribution on $\hat{\Omega}$ for $M = N = 4$, and Lagrangian basis functions for $M = N = 10$: $\pi_{10}\pi_2$, $\pi_2\pi_9$, and $\pi_3\pi_4$.

Deville, M.O, Fischer, P.F, and Mund E.H. High order methods for incompressible fluid flow. Cambridge University Press, 2002

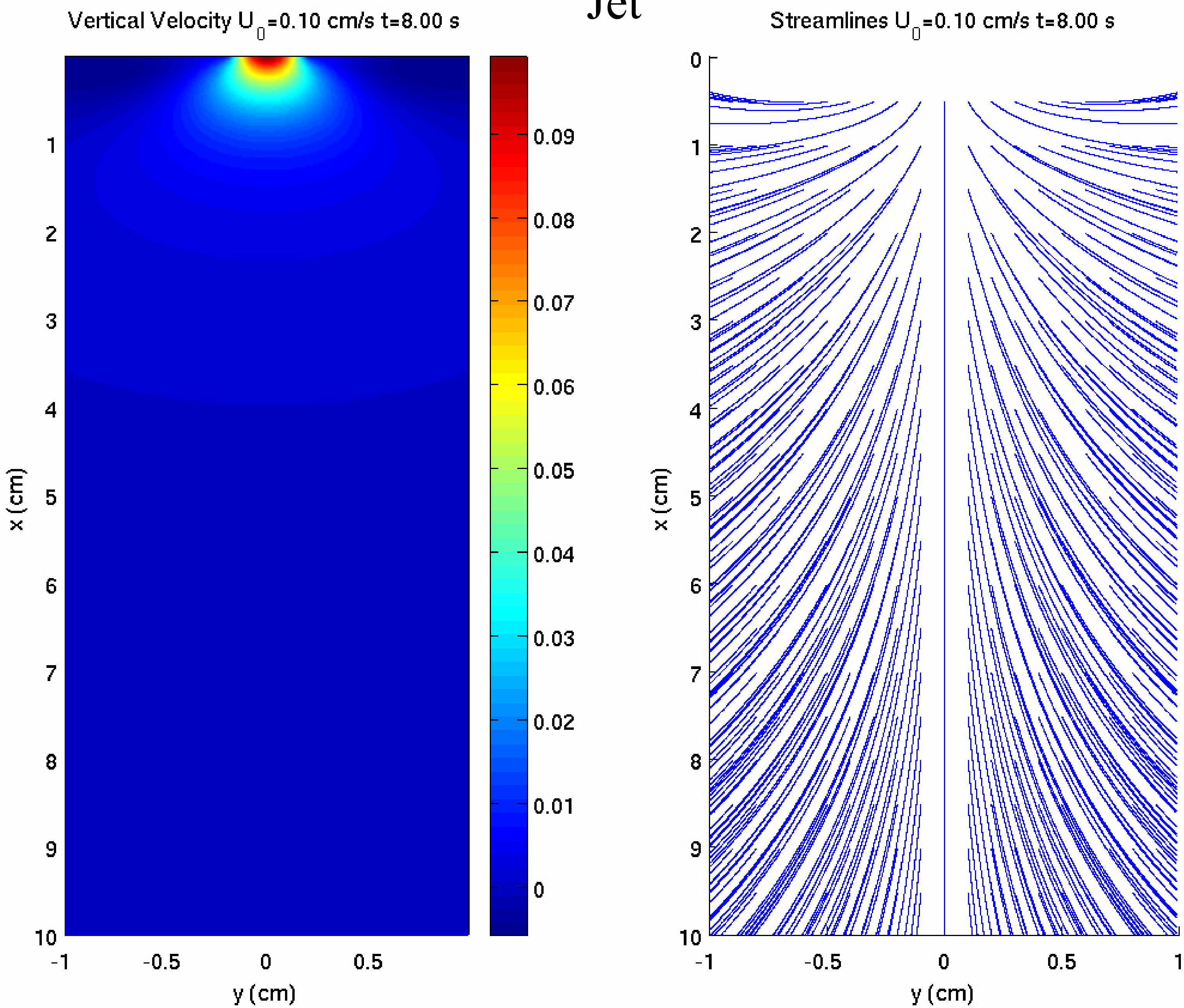
Tensor Product Matrices

Operators also take a tensor product form,

$$\mathbf{D}_x = \mathbf{I} \otimes \hat{\mathbf{D}}_\mathbf{x} = \begin{bmatrix} \hat{\mathbf{D}}_\mathbf{x} & & & & \\ & \hat{\mathbf{D}}_\mathbf{x} & & & \\ & & \ddots & & \\ & & & \hat{\mathbf{D}}_\mathbf{x} & \\ & & & & \hat{\mathbf{D}}_\mathbf{x} \end{bmatrix}$$

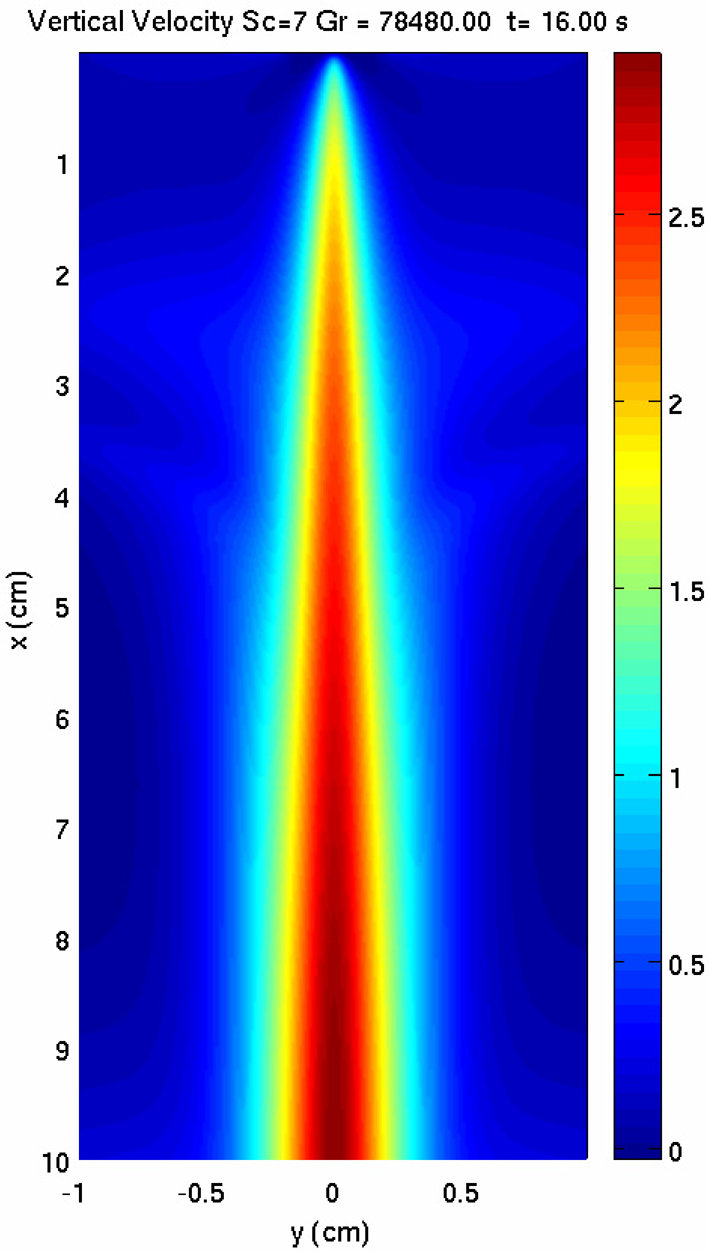
Evaluation of the action of operator is efficient $O\left(n^{d+1}\right)$

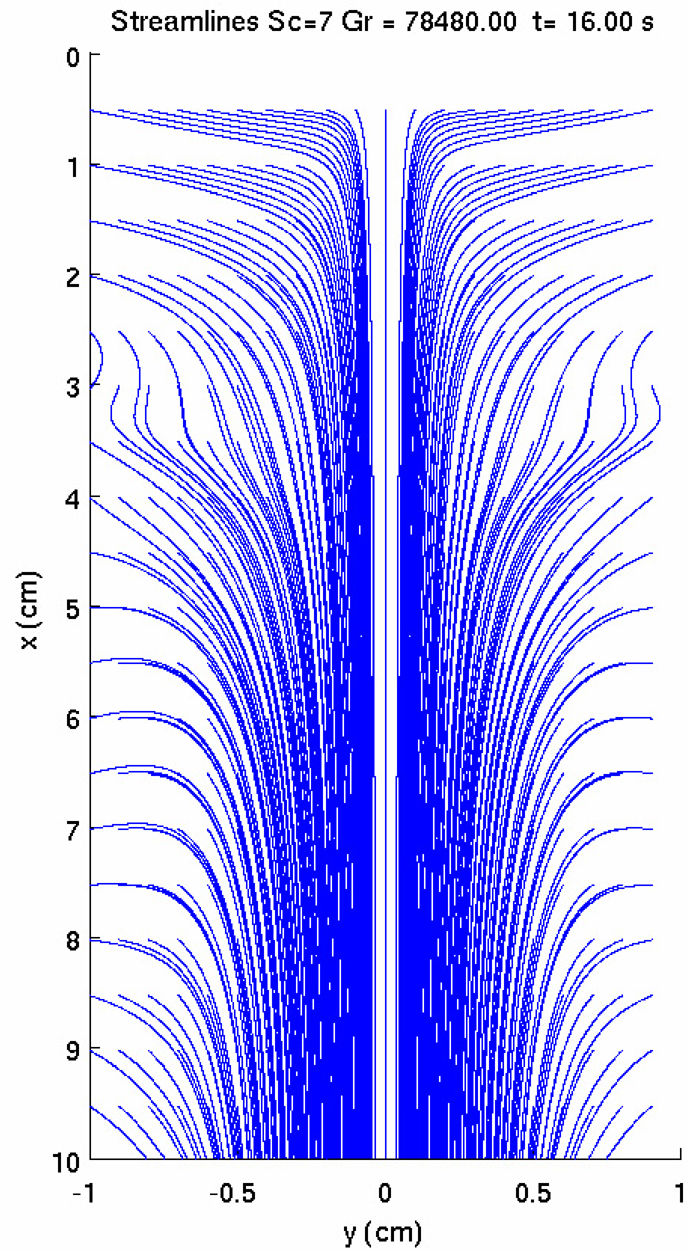
Jet



Vertical Velocity U0=0.10 cm/s t=8.00 s
x (cm)
y (cm)
-1
-0.5
0
0.5
1
2
3
4
5
6
7
8
9
10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Streamlines U0=0.10 cm/s t=8.00 s

Plume

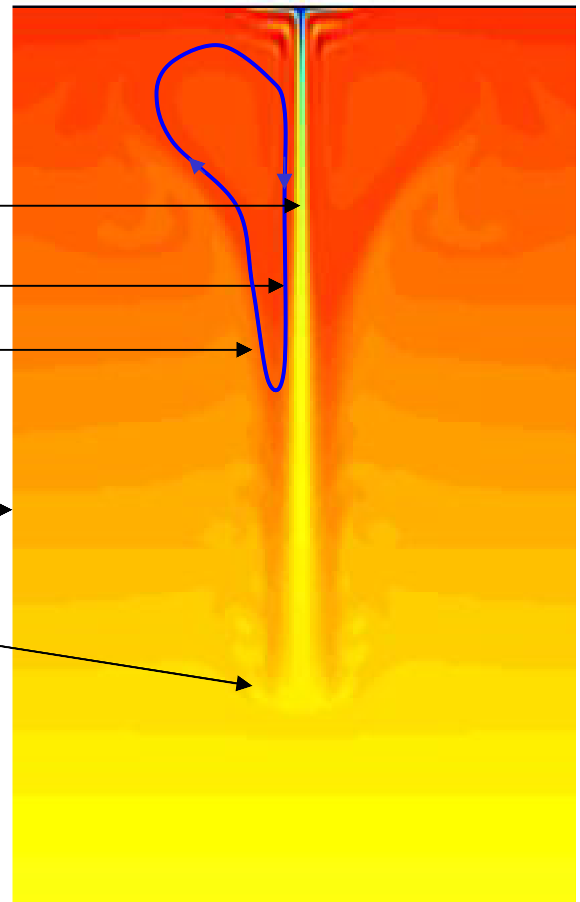

Vertical Velocity Sc=7 Gr = 78480.00 t= 16.00 s
x (cm)
y (cm)
2.5
2
1.5
1
0.5
0
-1
-0.5
0
0.5
1
2
3
4
5
6
7
8
9
10


Streamlines Sc=7 Gr = 78480.00 t= 16.00 s
x (cm)
y (cm)
0
1
2
3
4
5
6
7
8
9
10
-1
-0.5
0
0.5
1

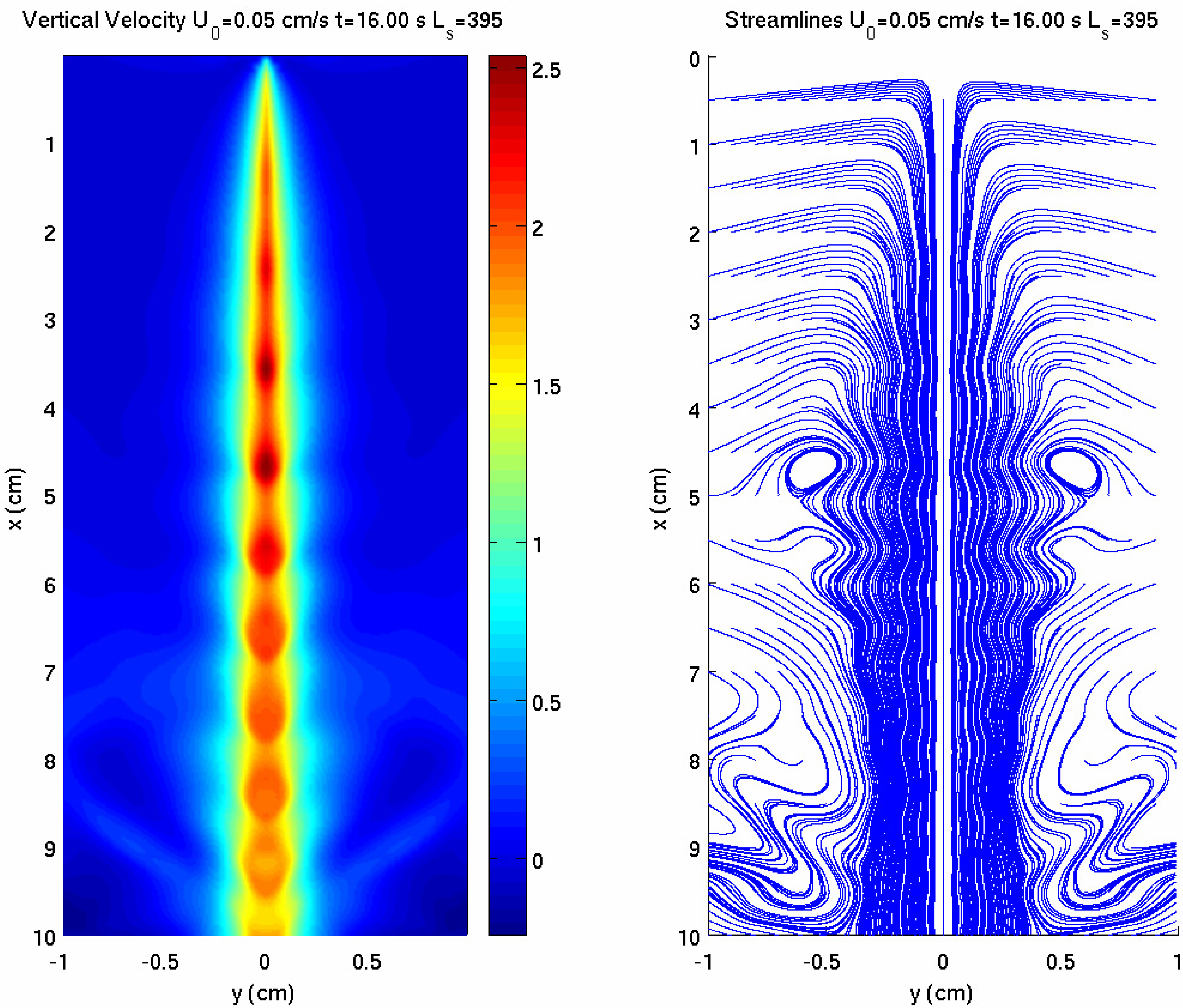
Flow zones for buoyant jets in stratified fluids

(Tenner and Gebhart ’71)

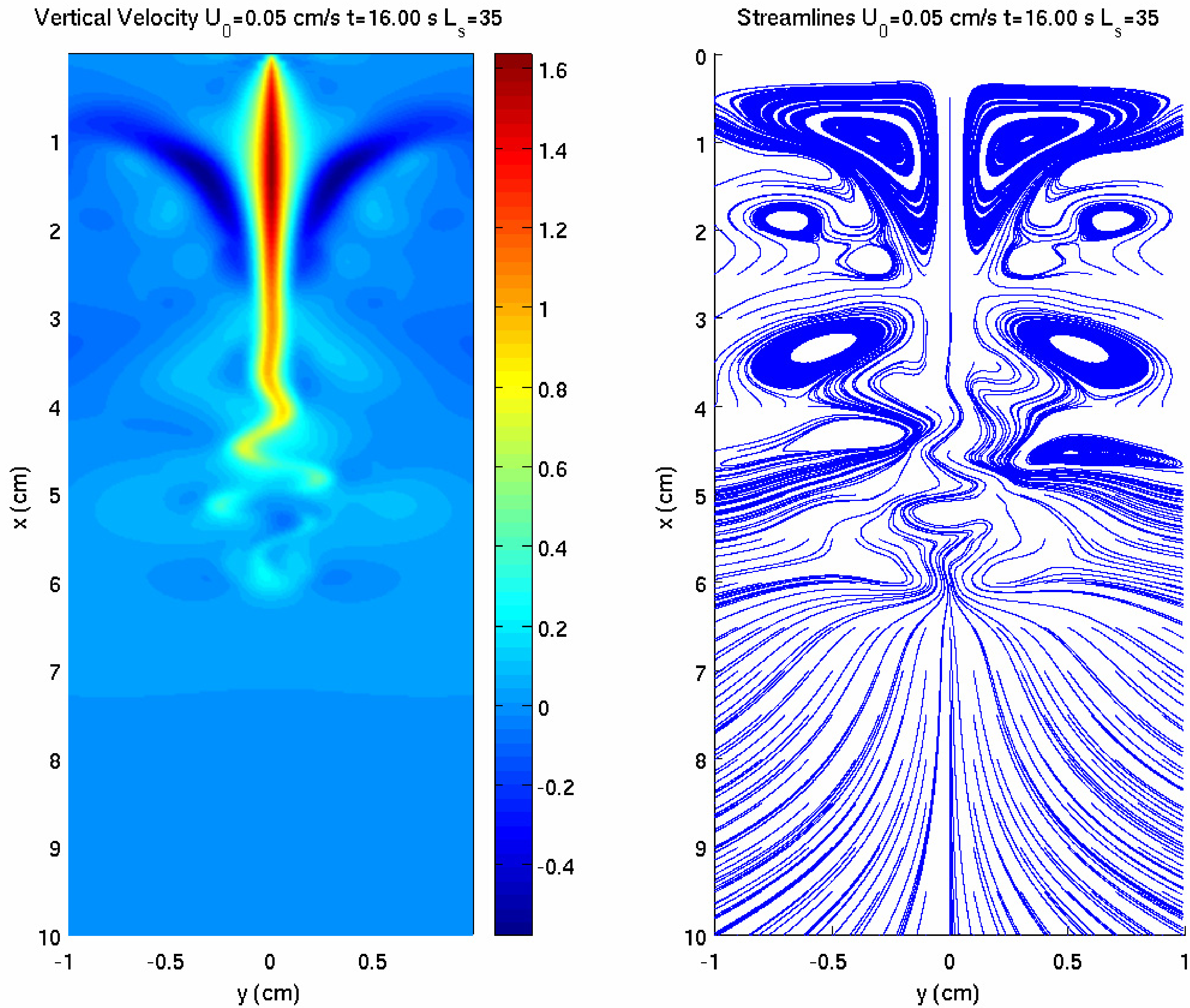
1. Jet core
2. Jet-cell interface
3. Toroidal cell
4. Shroud
5. Far Field
6. Cloud
7. Shroud Base



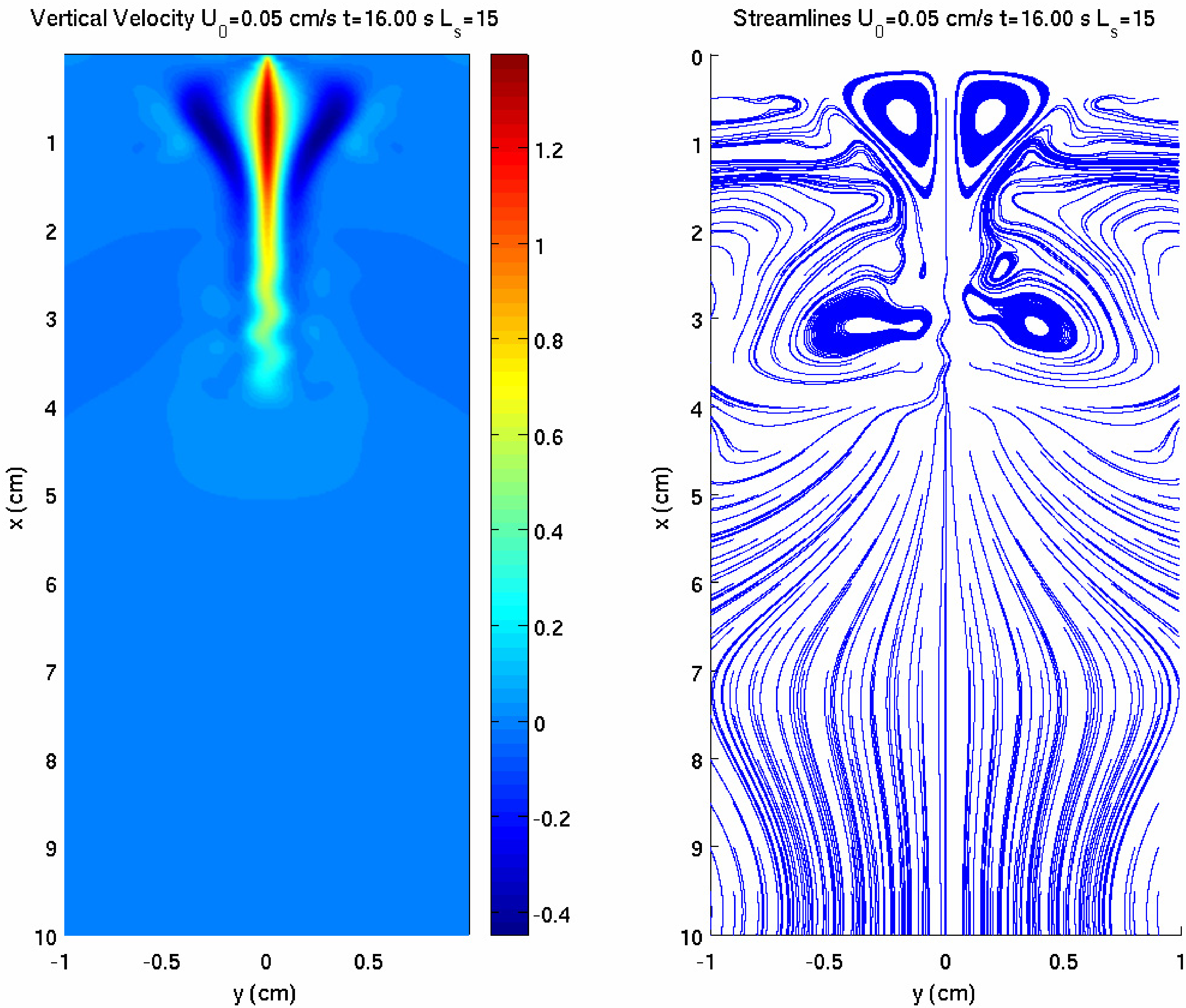
A. R. Tenner and Gebhart, B. “Laminar and axysmmetric jets in a stably stratified environment” Int. J. Heat and Mass Transfer, 14, 1971



Vertical Velocity U_0=0.05 cm/s t=16.00 s L_s=395
Streamlines U_0=0.05 cm/s t=16.00 s L_s=395
x (cm)
y (cm)
0
1
2
3
4
5
6
7
8
9
10
-1
-0.5
0
0.5
1
2.5
2
1.5
1
0.5
0

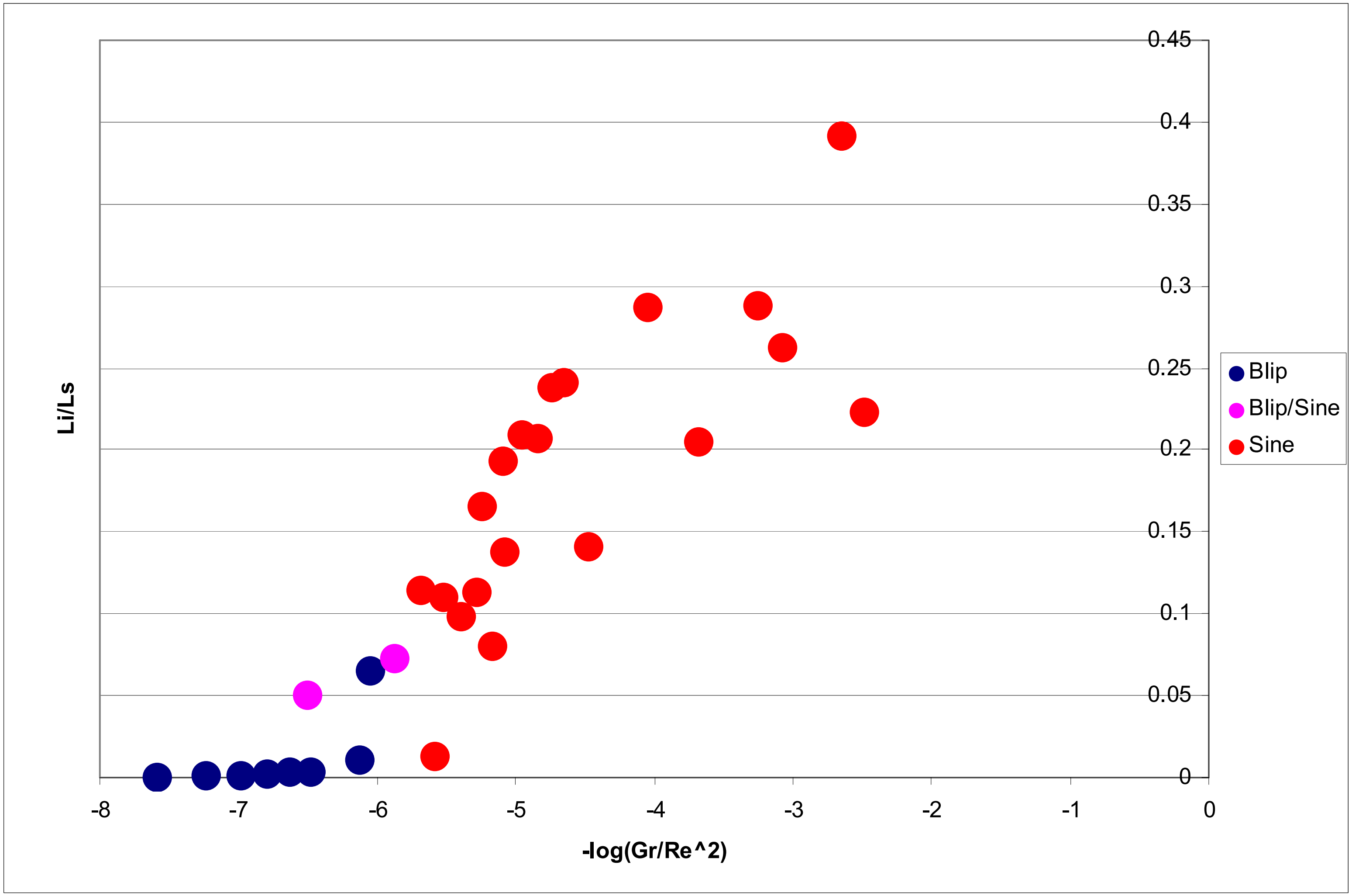


Vertical Velocity U0=0.05 cm/s t=16.00 s Ls=35
Streamlines U0=0.05 cm/s t=16.00 s Ls=35
x (cm)
y (cm)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4

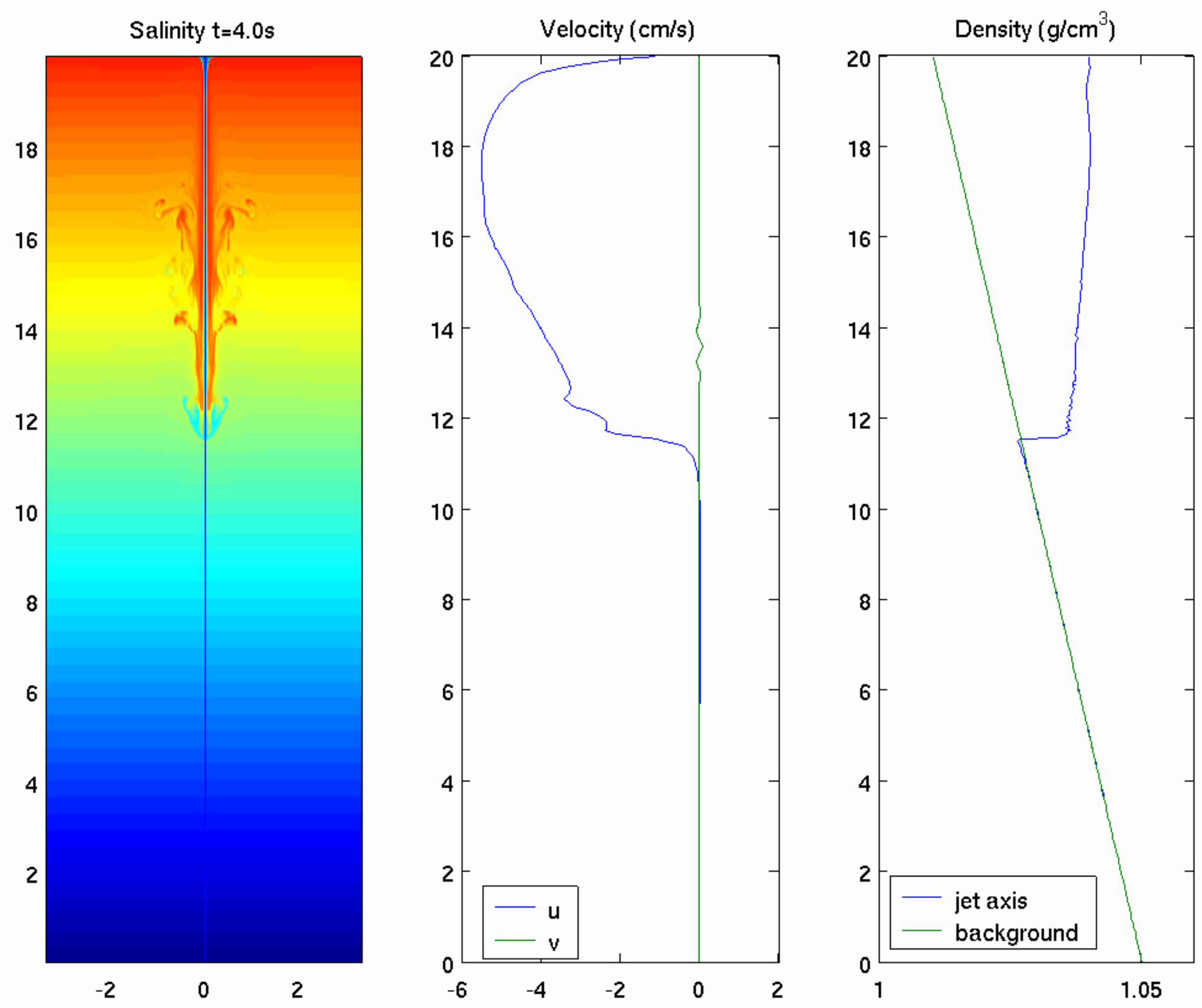


Vertical Velocity U0=0.05 cm/s t=16.00 s Ls=15
Streamlines U0=0.05 cm/s t=16.00 s Ls=15
x (cm)
y (cm)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4

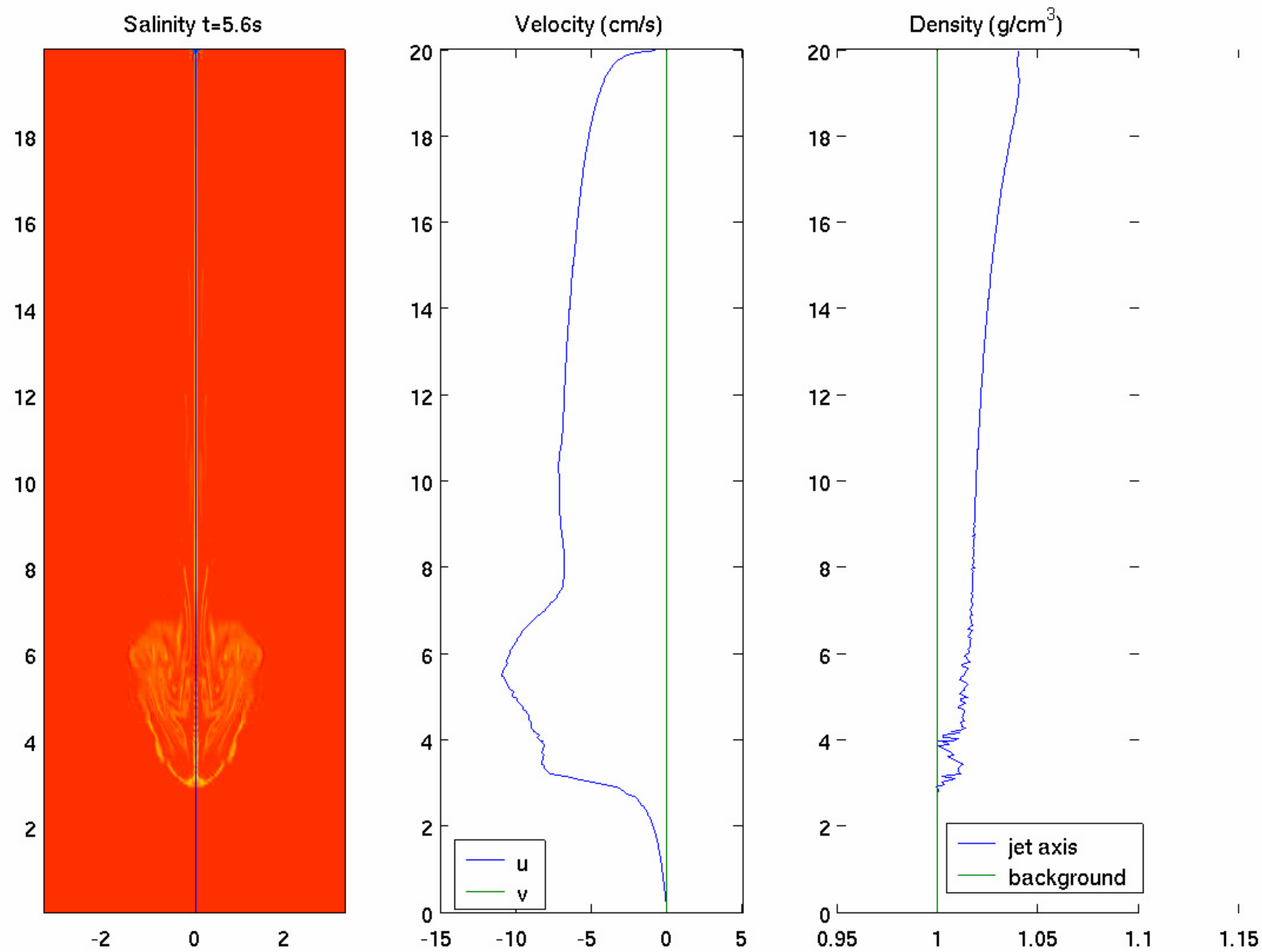
Stability



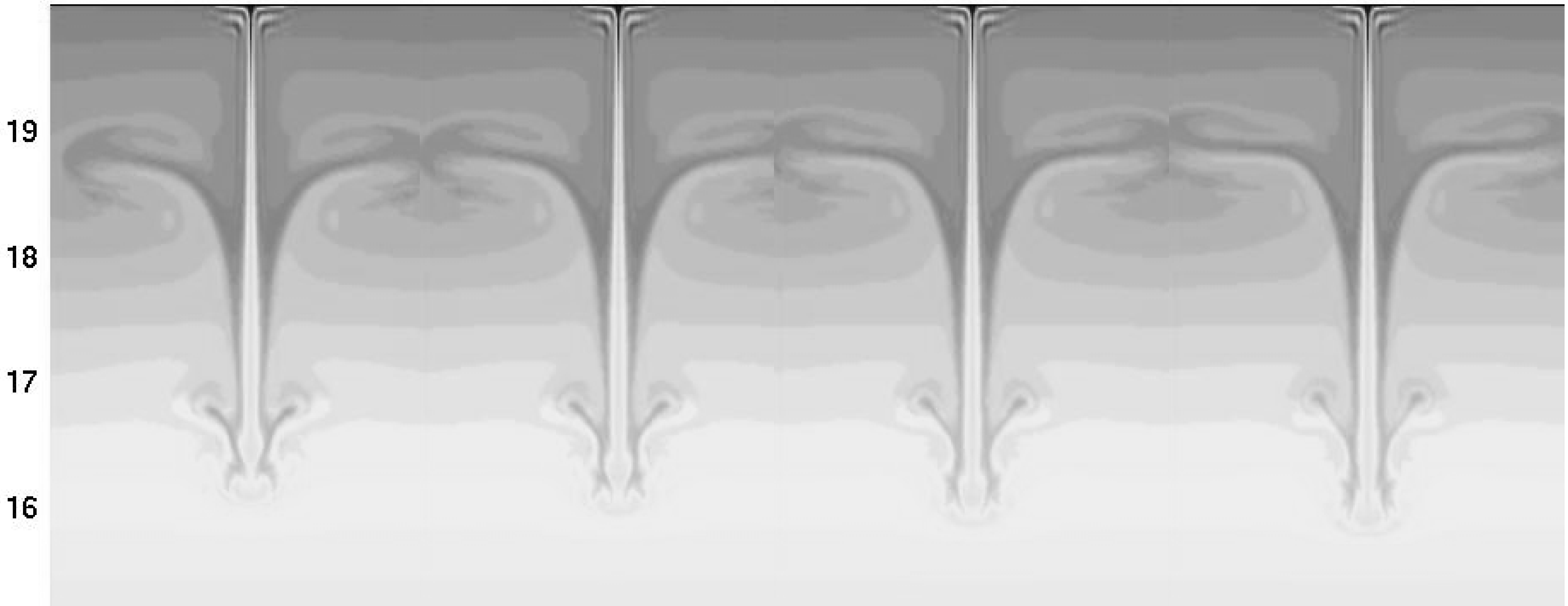
$$\mathrm{Gr} = \frac{g\beta\left(S_j - S_\infty(x=0)\right)L_s^3}{\nu^2} \quad \mathrm{Re} = \frac{U_0 L_s}{\nu} \quad L_s = \frac{S_j - S_\infty(x=0)}{\frac{\partial S_\infty}{\partial x}}$$

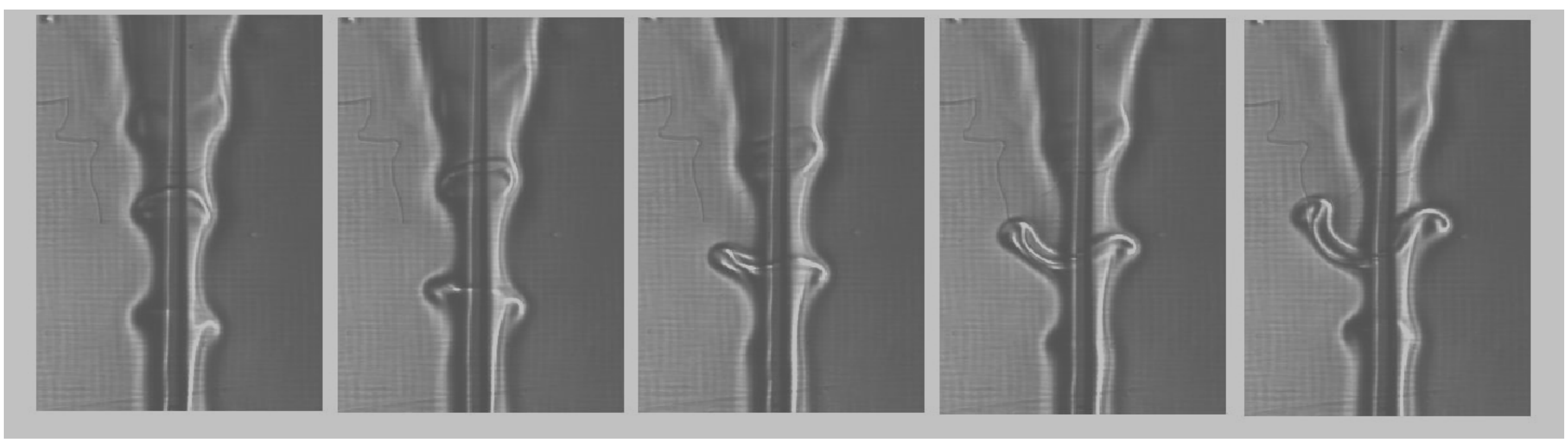


Salinity t=4.0s
Velocity (cm/s)
Density (g/cm^3)
u
v
jet axis
background


Salinity t=5.6s
Velocity (cm/s)
Density (g/cm^3)
u
v
jet axis
background

Conduit Instability





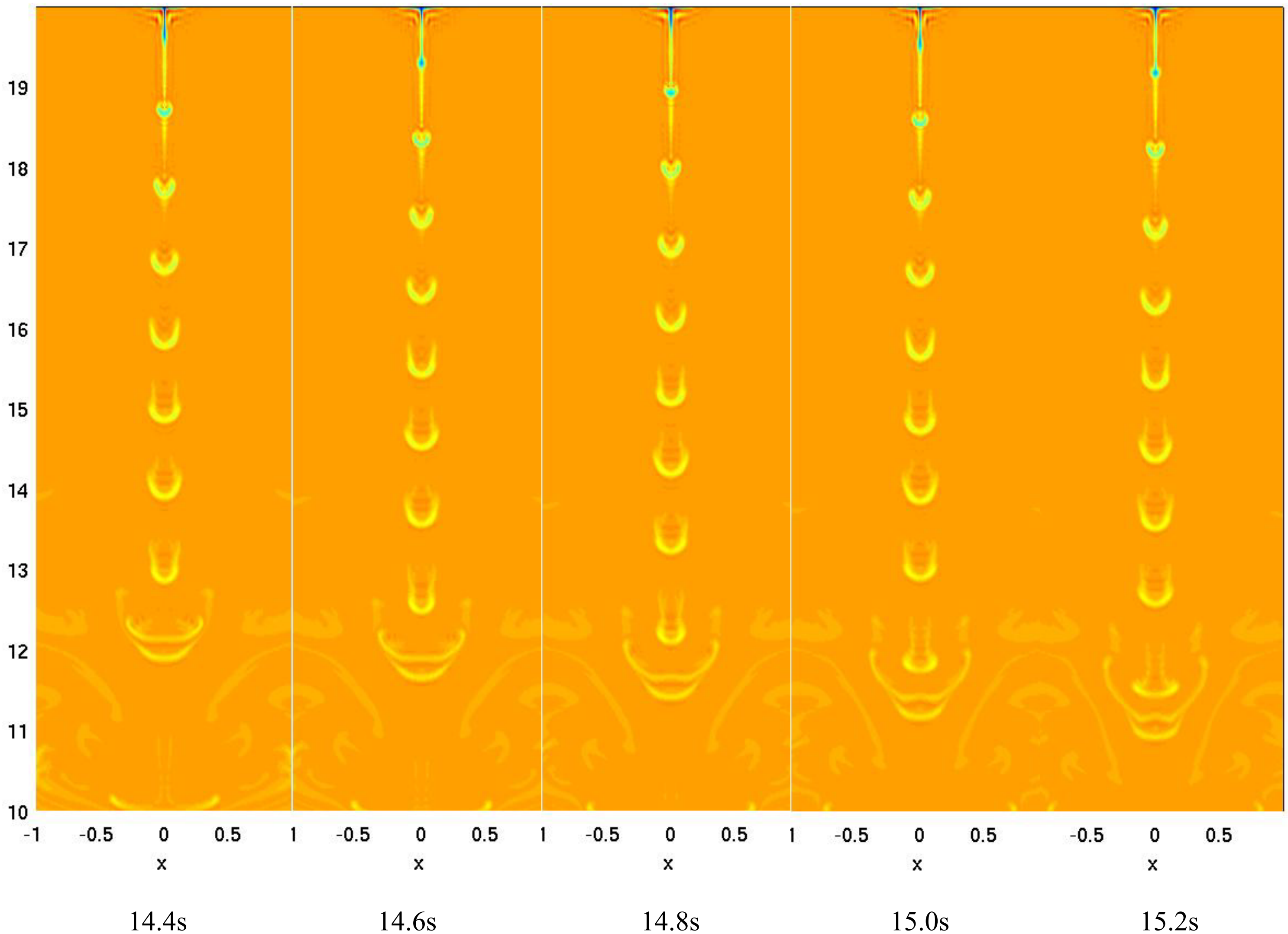
Salinity.mov

Blips

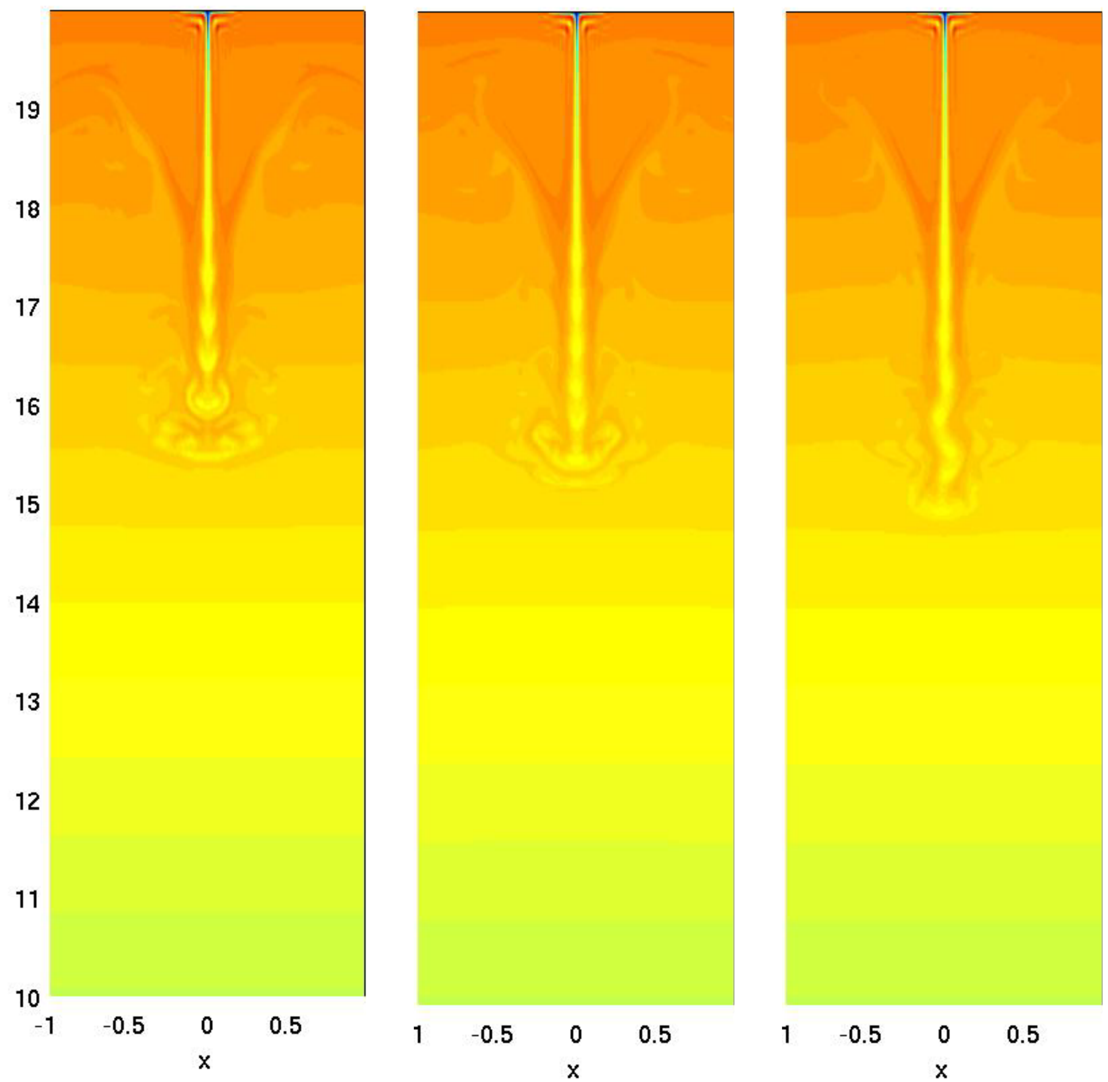
$SC = 100$

$U_0 = 0.04\ cm/s$

$\frac{1}{\rho_0}\frac{\partial \rho}{\partial x} = 0.001\ cm^{-1}$



Transitory Blips/Sines



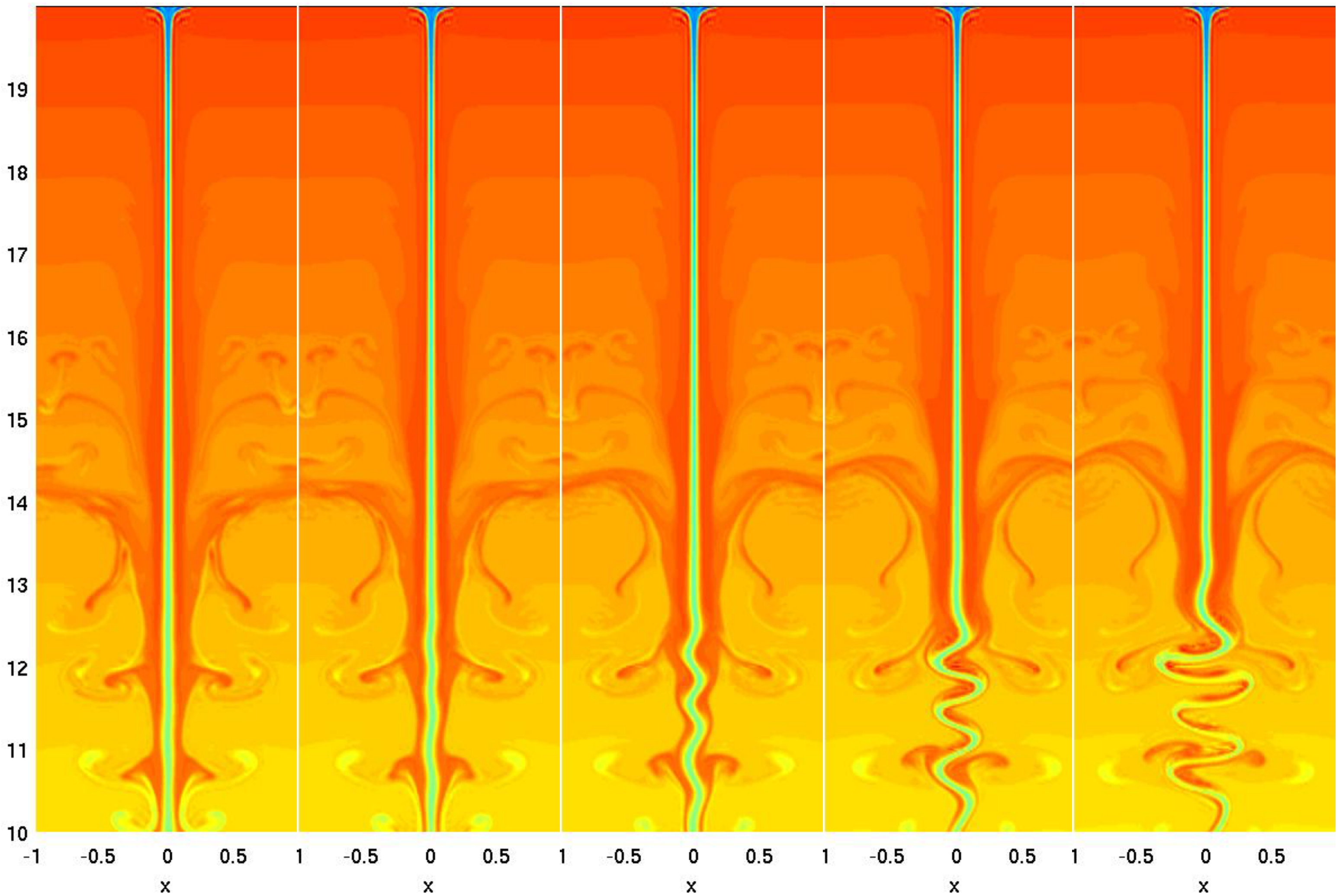
Buckling Instability

Salinity.mov

$SC = 100$

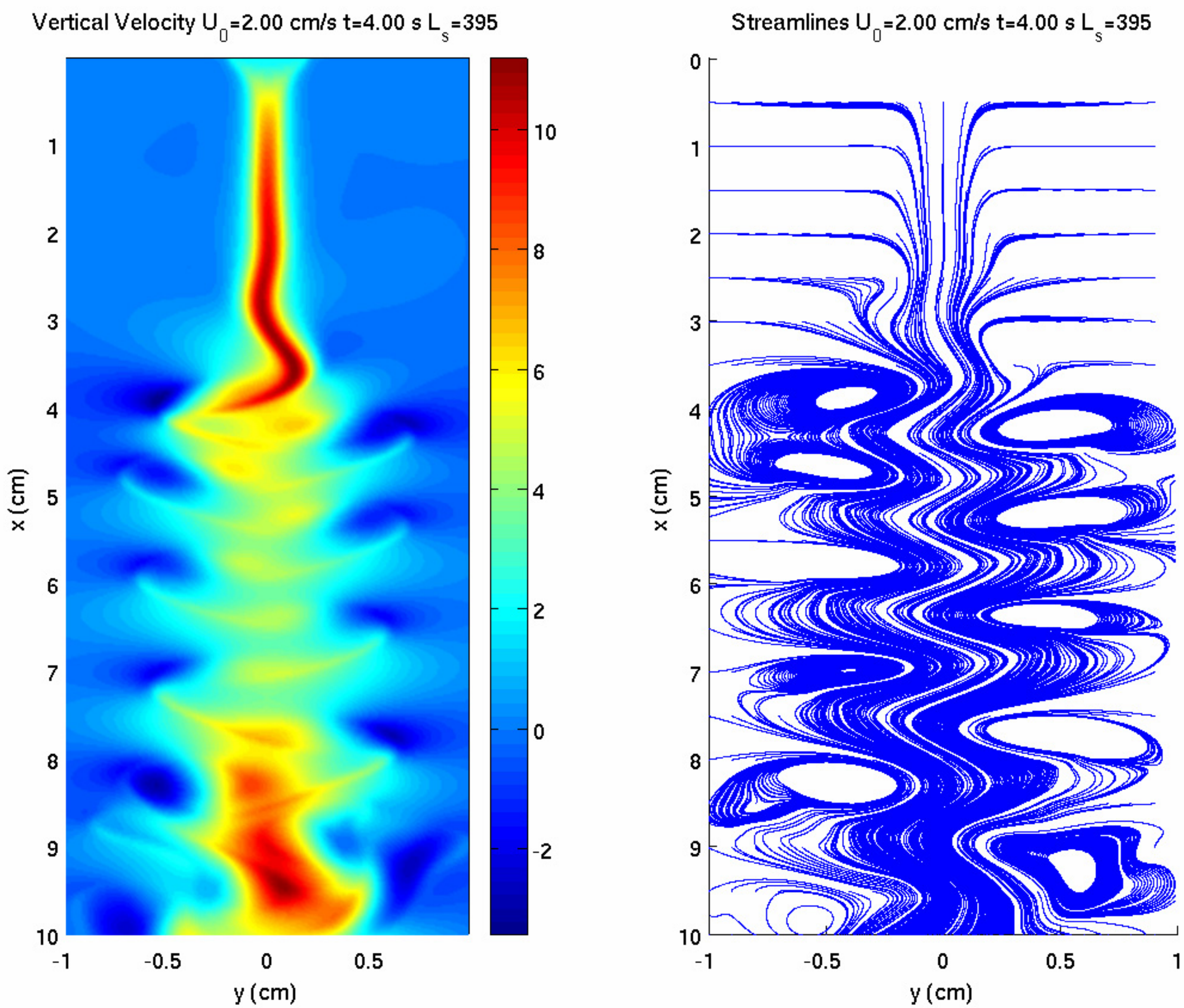
$U_0 = 0.50\ cm/s$

$\frac{1}{\rho_0}\frac{\partial \rho}{\partial x} = 0.001\ cm^{-1}$

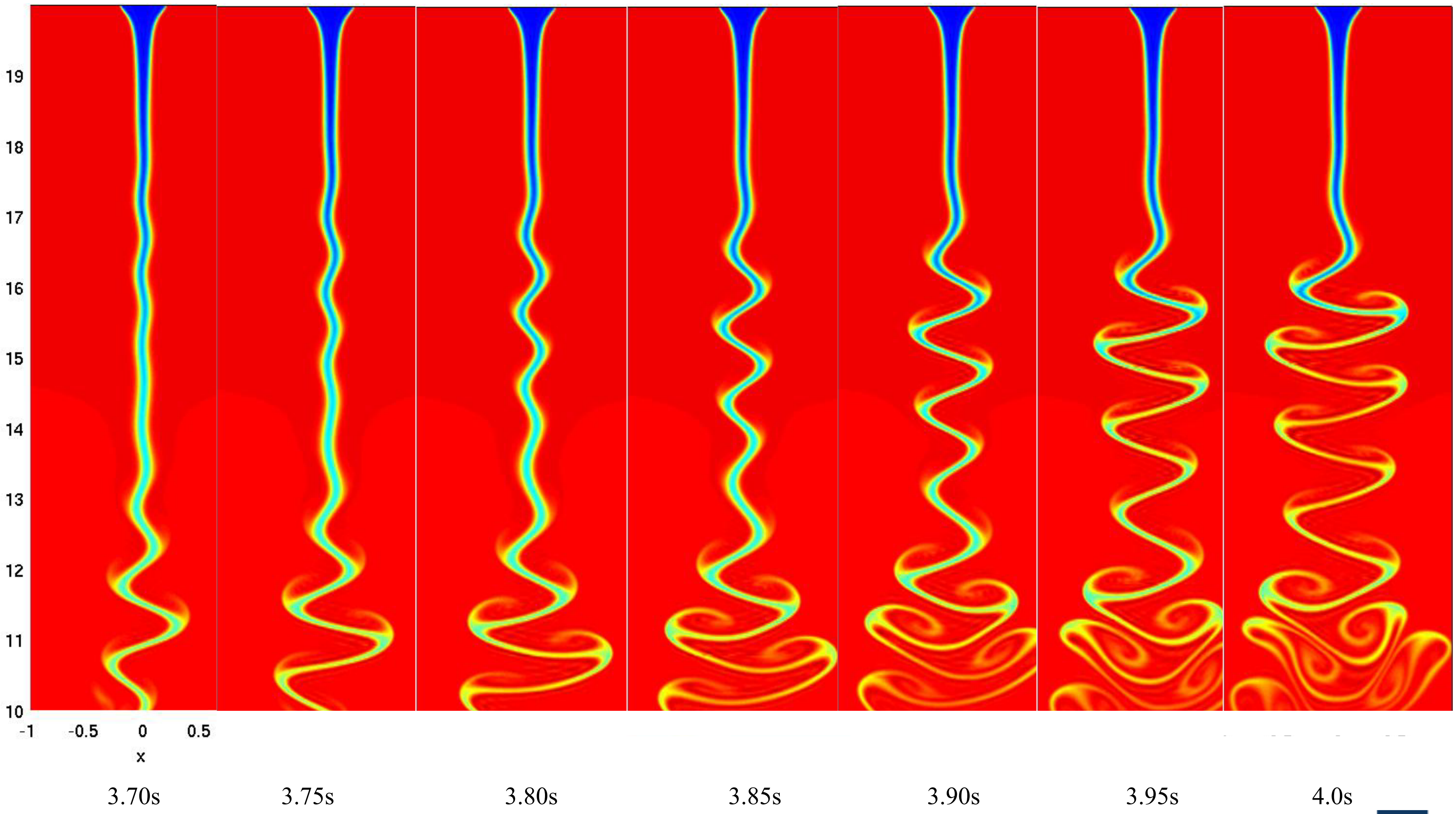


5.6s 5.8s 6.0s 6.2s 6.4s

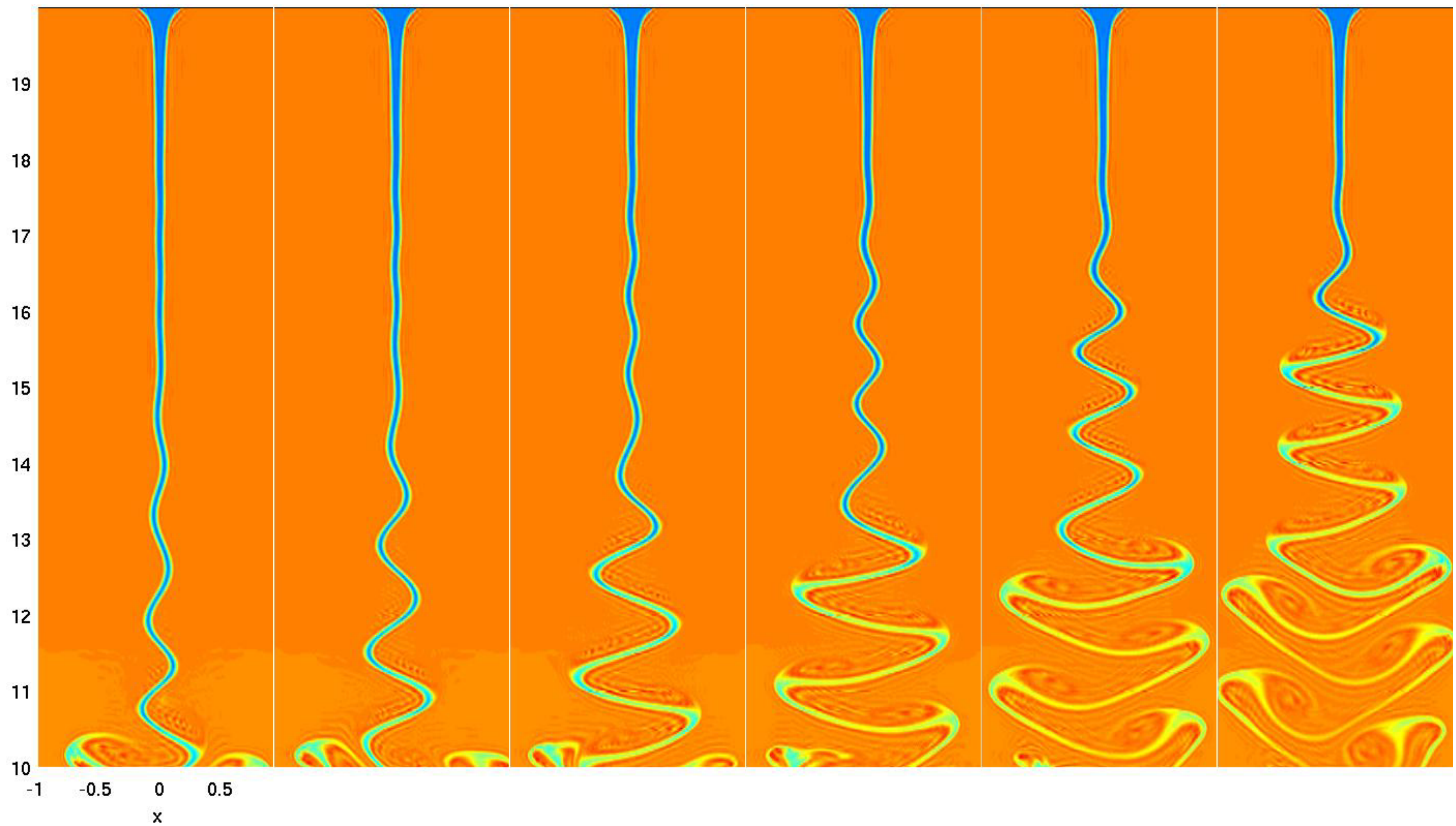
Vortex Street (Sc=7)



Planar Coiling (Sc=7)



Planar Coiling (Sc=100)



4.2s 4.3s 4.4s 4.5s 4.6s 4.7s

Ongoing Research

- Numerical bifurcation analysis
- The stability impact of the evolution of velocity profile along the jet axis.
- Analytical/reduced models
- Interaction of the jet buckling instability with conduit.
- Multiple jets interactions.

Acknowledgements

- Krell Institute, DOE Computational Science Graduate Fellowship
- Dr. Paul Fischer, Argonne National Laboratory