Linear Solution Preservation and Diffusive Solutions for $S_N$ Radiation Transport

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Radiation Modeling and Simulation

- Radiation Particles
  - Photons in thermal radiation
  - Neutrons and Photons in nuclear reactors
  - Neutrons, Photons, and Charged Particles in medical physics
  - Above plus Heavies in aerospace applications
  - Above plus Neutrinos in supernovae

- Particle average behavior modeled by Boltzmann equation
  - Particle simulations by Monte Carlo
  - Direct simulation by deterministic method

- Equations have both hyperbolic and elliptic qualities
Basic Transport

Radiation Scattered to Direction $\Omega'$

Isotropic Emission with Density $Q/4\pi$

$c = \sigma_s / \sigma$

$\lambda = 1 / \sigma$

Transmitted Radiation

Incident Radiation in Direction $\Omega$
Basic Transport

Absorber

Boundary Layer

Thick Diffusive Object

Thin Streaming Region

Thick Diffusive Object
HEDP Codes

Alegra-HEDP
(IMC/FLD)

Kull
(IMC/UCB)

Hydra
(IMC/FLD)
Steady-State Transport Equation

\[ \hat{\Omega} \cdot \nabla \psi + \sigma \psi = \frac{\sigma_s}{4\pi} \int_{4\pi} \psi \, d\Omega' + \frac{Q}{4\pi} \]

where

\[ \psi = \psi(r, \hat{\Omega}) \] is the radiation energy density, \\
\[ \hat{\Omega} = \text{direction of travel unit vector}, \]
\[ \sigma = \text{opacity (1/length)}, \]
\[ \sigma_s = \text{effective scattering cross section (1/length)}, \]
\[ Q = \text{volume source}. \]
After operator-splitting,

\[
\frac{\partial (\rho e)}{\partial t} = \int_{4\pi} d\Omega \sigma (\psi - B)
\]

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \hat{\Omega} \cdot \nabla \right) \psi = -\sigma \psi + \sigma B(T)
\]
Example – Radiative Transfer

After operator-splitting,

\[
\frac{\partial (\rho e)}{\partial t} = \left[ C_v \frac{\partial T}{\partial t} = \left( C_v \frac{\partial T}{\partial B} \right) \frac{\partial B}{\partial t} \right] = \int_{4\pi} d\Omega \sigma (\psi - B)
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\]

Linearized semi-implicit substitution gives

\[
\left[ \hat{\Omega} \cdot \nabla + \left( \sigma + \frac{1}{c\Delta t} \right) \right] \psi = \left( \frac{1}{1 + \frac{C_v}{4\pi \sigma \Delta t \frac{\partial B}{\partial T}}} \right) \frac{\sigma}{4\pi} \int d\Omega' \psi + q.
\]
Asymptotic Diffusion Limit

Scattering dominates, and the transport equation

\[ \epsilon \hat{\Omega} \cdot \nabla \psi + \sigma \psi = \frac{\sigma - \epsilon^2 \sigma_a}{4\pi} \int d\Omega' \psi + \epsilon^2 \frac{Q}{4\pi}, \]

as \( \epsilon \to 0 \), limits to the diffusion equation

\[ -\nabla \cdot \frac{1}{3\sigma} \nabla \psi^{(0)} + \sigma_a \psi^{(0)} = Q \]

or using Fick’s Law,

\[ \nabla \cdot \mathbf{F}^{(1)} + \sigma_a \psi^{(0)} = Q \]

where

\[ \mathbf{F}^{(1)} = -D \nabla \psi^{(0)} = \int_{4\pi} \hat{\Omega} \psi^{(1)} d\Omega' \]

\[ D = 1/3\sigma \]
Transport $\rightarrow$ Diffusion in diffusion limit, $\psi^{(1)}$ is linear-in-angle.

Transport and Diffusion equations share certain low-order polynomial solutions, such as the linear solution ($\psi = x - \mu/\sigma, \phi = x$).

These are not necessarily related. We show in the diffusion limit that they are crucially linked.
Importance of Diffusion Limit

In diffusive regions, the radiation density varies an $O(1)$ amount in a diffusion length, $\frac{1}{\sqrt{\sigma_\sigma_a}}$, so the mesh spacing should only have to resolve that scale.

In transport dominated regions, the radiation density can vary an $O(1)$ amount over a mean free path, $\lambda = \frac{1}{\sigma}$, but in a diffusive region this becomes $\frac{\epsilon}{\sigma}$.

If a differencing scheme is designed to be accurate only in the transport regime, excessively fine meshes will be required to obtain accurate solutions in diffusive regions.

If a differencing scheme scales asymptotically to a discretized version of the diffusion equation, then we can use thick cells accurately in diffusive regions.
## History

### 1987 Larsen, Morel, and Miller → Discrete Asymptotic Analysis

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<td>Hanshaw &amp; Larsen, MMB</td>
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### 2003 Larsen → Polynomial Solutions Paper
Skewed Kershaw Grids

\[ \xi = 0.5 \quad \xi = 0.2 \quad \xi = 0.01 \]
Diffusive Hump Problem

() MMB, $\xi=0.5$

() MMB, $\xi=0.2$

() MMB, $\xi=0.01$

() UCB, $\xi=0.5$

() UCB, $\xi=0.2$

() UCB, $\xi=0.01$
Define

\[ \psi_x = (x - \mu/\sigma)/4\pi \]

Then

\[ F_{exact}(\psi_x) = -\hat{x}/3\sigma = -D\hat{x} \]

Compute approximate \( \psi_x \) and \( \psi_y \) solutions to linear problems, then calculate

\[ D_{eff} = - \begin{bmatrix} F_x(\psi_x) & F_x(\psi_y) \\ F_y(\psi_x) & F_y(\psi_y) \end{bmatrix} \]

Computational alternative to Larsen’s \( D(h) \) analysis.
UCB Results for $\xi = 0.1$

$D_{xx}$

$D_{yy}$

$\phi$
Adams “Hard” Problem

Grid

UCB

MMB
Interpretation

- Stand-Alone Diffusion Discretization
  - Does not have to preserve linear solutions, which may be desirable (because of loss of symmetry).
  - E.g. Morel & Shashkov’s local support operators methods.

- Transport Diffusion-Limit Diffusion Discretization
  - Failure to preserve linear-solutions yields gross changes in flux.
  - This stems from the angular dependence of the linear solution.
  - $D_{eff}$ analysis reveals this is the dominant cause of the diffusive solution degradation.

- Ideally, a transport solution will always preserves the linear solution; in the diffusion limit, this is a must!
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