Linear Solution Preservation and Diffusive Solutions for S_N Radiation Transport

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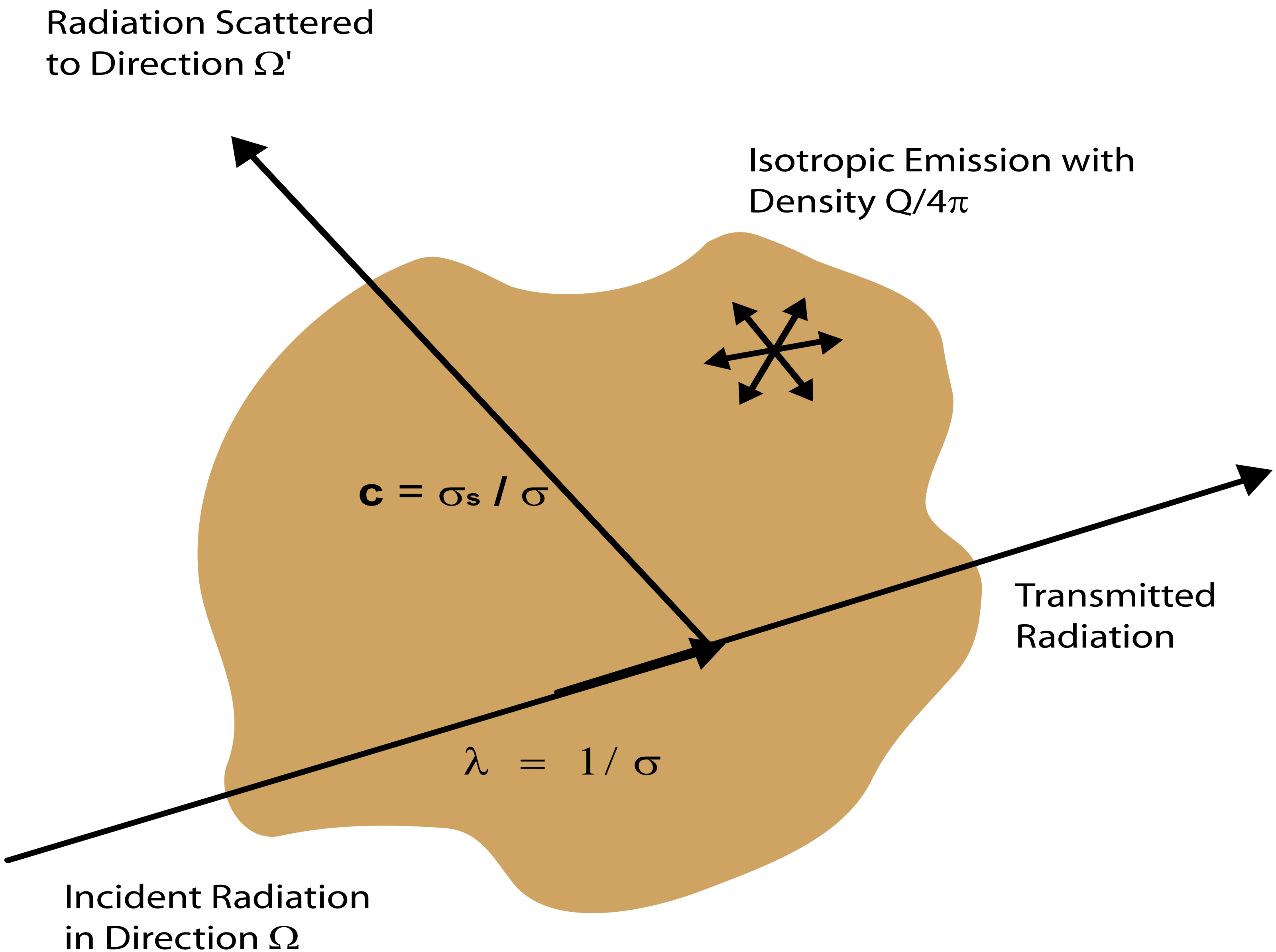
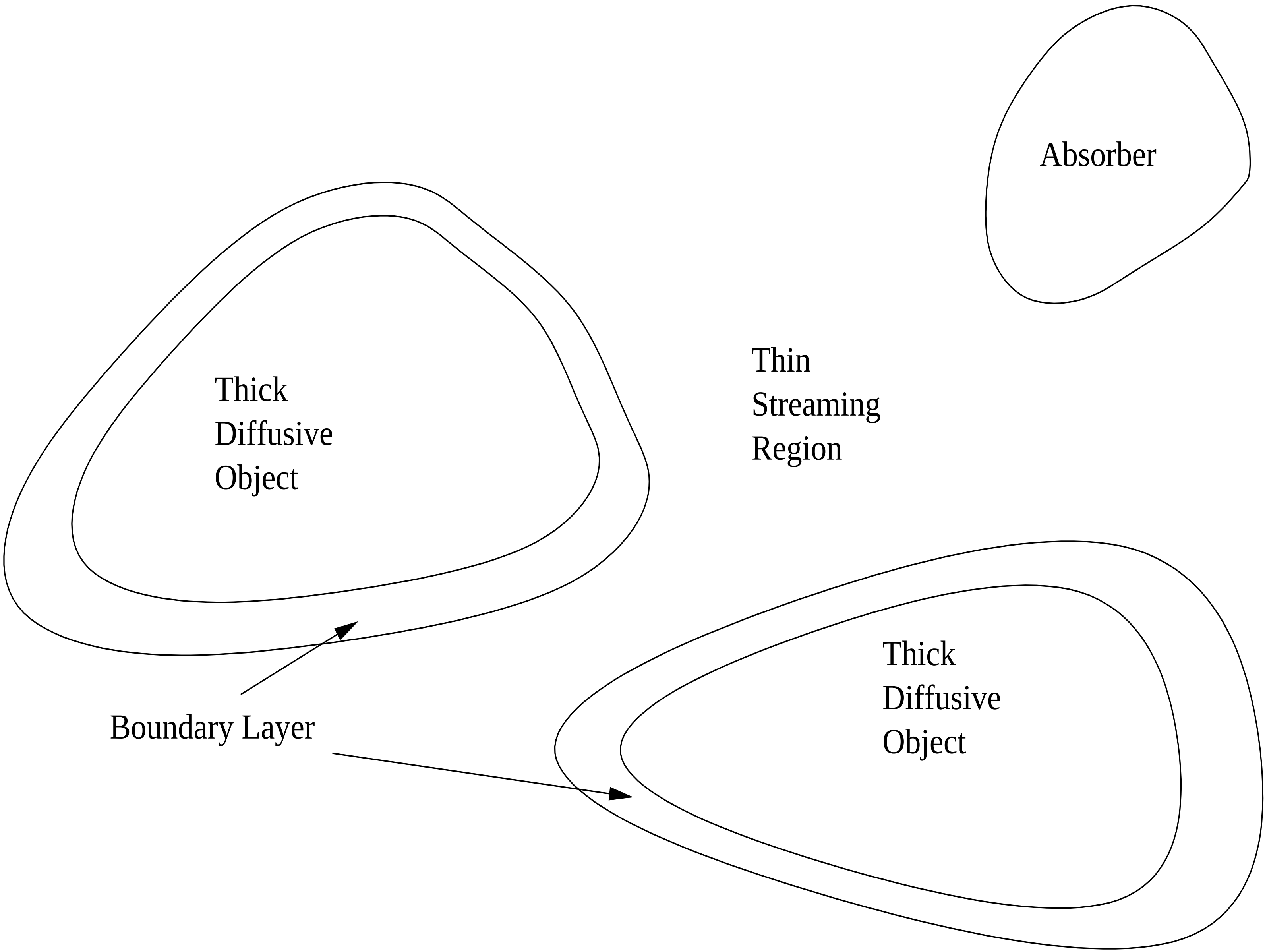
Radiation Modeling and Simulation

- Radiation Particles
  - Photons in thermal radiation
  - Neutrons and Photons in nuclear reactors
  - Neutrons, Photons, and Charged Particles in medical physics
  - Above plus Heavies in aerospace applications
  - Above plus Neutrinos in supernovae
- Particle average behavior modeled by Boltzmann equation
  - Particle simulations by Monte Carlo
  - Direct simulation by deterministic method
- Equations have both hyperbolic and elliptic qualities

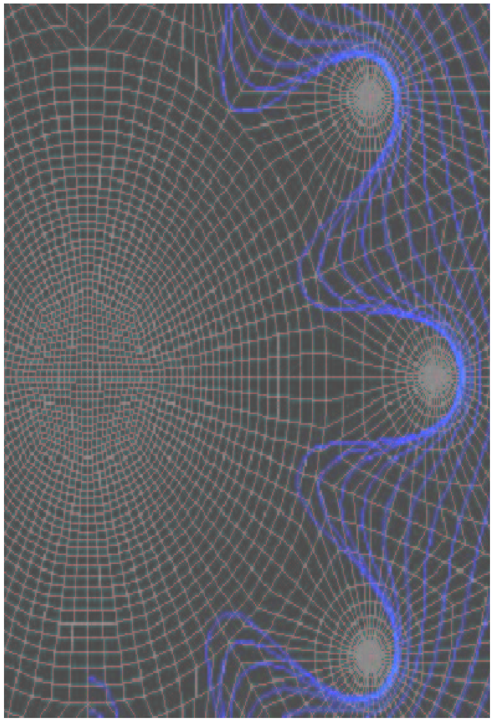
Basic Transport

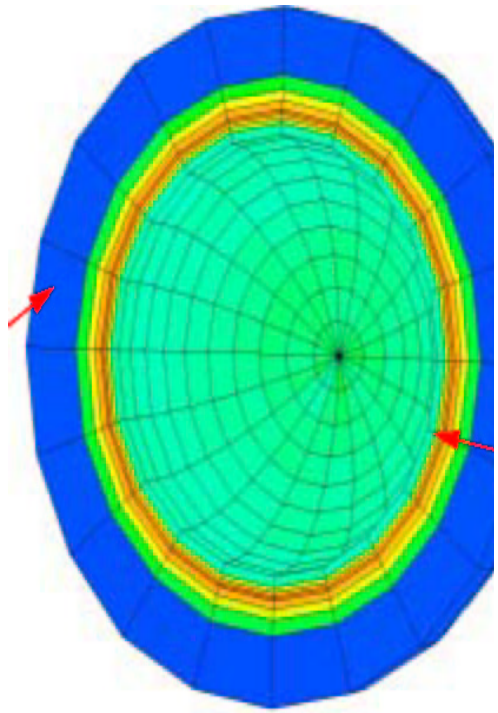


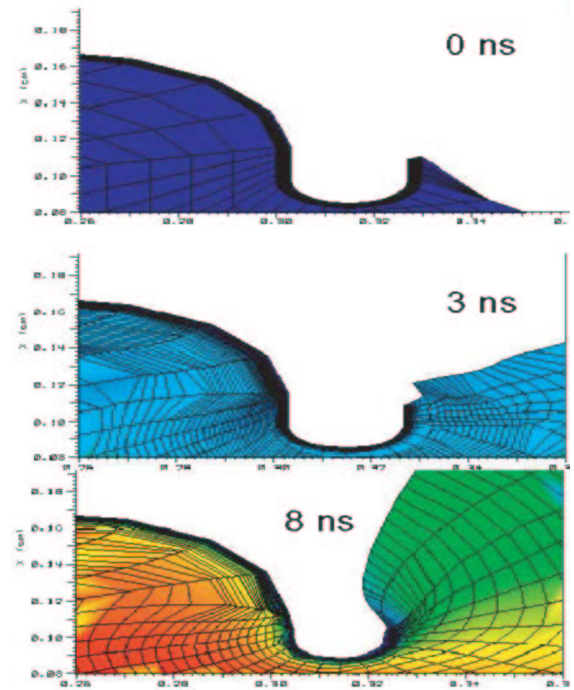
Basic Transport



HEDP Codes







Alegra-HEDP (IMC/FLD)

Kull (IMC/UCB)

Hydra (IMC/FLD)

Steady-State Transport Equation

$$\hat{\mathbf{\Omega}} \cdot \nabla\psi + \sigma\psi = \frac{\sigma_s}{4\pi}\int_{4\pi}\psi \, d\Omega' + \frac{Q}{4\pi}$$

where

$\psi = \psi(\mathbf{r}, \hat{\mathbf{\Omega}})$ is the radiation energy density,

$\hat{\mathbf{\Omega}} =$ direction of travel unit vector,

$\sigma =$ opacity (1/length),

$\sigma_s =$ effective scattering cross section (1/length),

$Q =$ volume source.

Example – Radiative Transfer

After operator-splitting,

$$\frac{\partial(\rho e)}{\partial t} = \int_{4\pi} d\Omega \, \sigma \, (\psi - B)$$

$$\left(\frac{1}{c} \frac{\partial}{\partial t} + \hat{\mathbf{\Omega}} \cdot \nabla \right) \psi = -\sigma\psi + \sigma B(T)$$

Example – Radiative Transfer

After operator-splitting,

$$\frac{\partial(\rho e)}{\partial t} = \left[C_v \frac{\partial T}{\partial t} = \left(C_v \frac{\partial T}{\partial B}\right) \frac{\partial B}{\partial t}\right] = \int_{4\pi} d\Omega \, \sigma \left(\psi - B\right)$$

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Linearized semi-implicit substitution gives

$$\left[\hat{\mathbf{\Omega}} \cdot \nabla + \left(\sigma + \frac{1}{c\Delta t}\right)\right]\psi = \left(\frac{1}{1 + \frac{C_v}{4\pi\sigma\Delta t \frac{\partial B}{\partial T}}}\right)\frac{\sigma}{4\pi}\int d\Omega' \psi + q.$$

Asymptotic Diffusion Limit

Scattering dominates, and the transport equation

$$\epsilon\hat{\mathbf{\Omega}} \cdot \nabla\psi + \sigma\psi = \frac{\sigma - \epsilon^2\sigma_a}{4\pi} \int d\Omega'\psi + \epsilon^2 \frac{Q}{4\pi},$$

as $\epsilon \to 0$, limits to the diffusion equation

$$-\nabla \cdot \frac{1}{3\sigma}\nabla\psi^{(0)} + \sigma_a\psi^{(0)} = Q$$

or using Fick's Law,

$$\nabla \cdot \mathbf{F}^{(1)} + \sigma_a\psi^{(0)} = Q$$

where

$$\mathbf{F}^{(1)} = -D\nabla\psi^{(0)} = \int_{4\pi} \hat{\mathbf{\Omega}}\psi^{(1)}\, d\Omega'$$

$$D = 1/3\sigma$$

Diffusion Limit and Linear Solutions

$$\hat{\mathbf{\Omega}} \cdot \nabla\psi + \sigma\psi = \frac{\sigma_s}{4\pi}\int \psi \, d\hat{\mathbf{\Omega}}' + \frac{Q}{4\pi},$$

$$-\nabla \cdot \frac{1}{3\sigma}\nabla\phi + \sigma_a\phi = Q.$$

- Transport $\longrightarrow$ Diffusion in diffusion limit, $\psi^{(1)}$ is linear-in-angle.
- Transport and Diffusion equations share certain low-order polynomial solutions, such as the linear solution $(\psi = x - \mu/\sigma, \quad \phi = x)$.
- These are not necessarily related. We show in the diffusion limit that they are crucially linked.

Importance of Diffusion Limit

- In diffusive regions, the radiation density varies an $O(1)$ amount in a diffusion length, $\frac{1}{\sqrt{\sigma\sigma_a}}$, so the mesh spacing should only have to resolve that scale.
- In transport dominated regions, the radiation density can vary an $O(1)$ amount over a mean free path, $\lambda = \frac{1}{\sigma}$, but in a diffusive region this becomes $\frac{\epsilon}{\sigma}$.
- If a differencing scheme is designed to be accurate only in the transport regime, excessively fine meshes will be required to obtain accurate solutions in diffusive regions.
- If a differencing scheme scales asymptotically to a discretized version of the diffusion equation, then we can use thick cells accurately in diffusive regions.

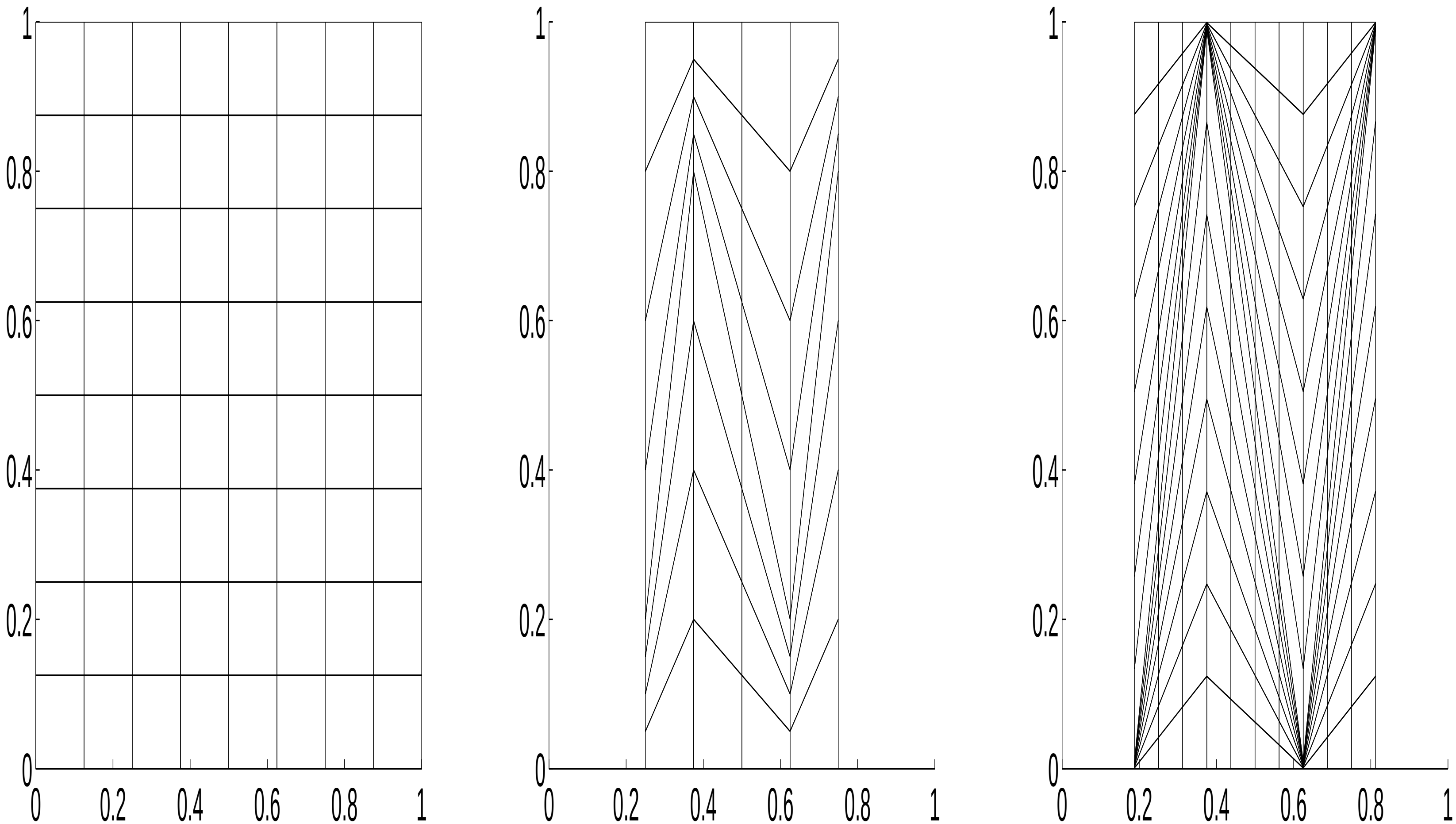
History

- 1987 Larsen, Morel, and Miller $\longrightarrow$ Discrete Asymptotic Analysis

Date	CB/DFEM		Multiple Balance	
1991	Adams,WLA	BLD	Morel/Larsen	MB2
	Adams	CB		
1997	Adams	UCB		
1998	Adams/Nowak	MG-UCB		
1999	Thompson	Tet-UCB		
2001	Waering et al.	Tri/Tet DFEM		
	Adams	DFEM		
2004	Davidson	Wachspress		
Today	H. Stone & Adams	PWLD	Hanshaw & Larsen	MMB

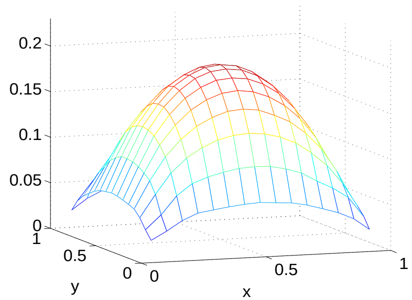
- 2003 Larsen $\longrightarrow$ Polynomial Solutions Paper

Skewed Kershaw Grids

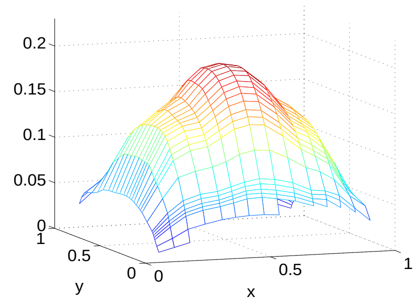


$\xi = 0.5$ $\xi = 0.2$ $\xi = 0.01$

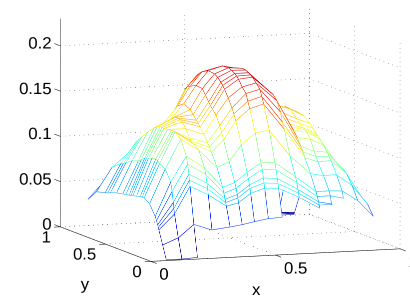
Diffusive Hump Problem



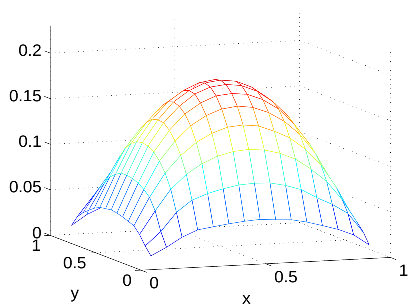
() MMB, ξ=0.5



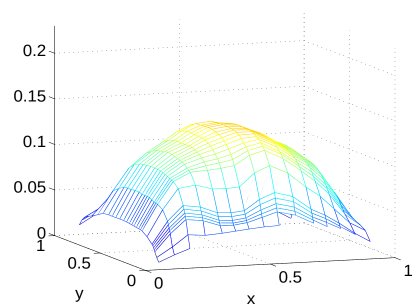
() MMB, ξ=0.2



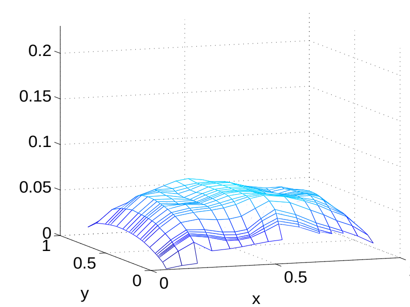
() MMB, ξ=0.01



() UCB, ξ=0.5



() UCB, ξ=0.2



() UCB, ξ=0.01

Effective Diffusion Tensor Analysis

Define

$$\psi_x = (x - \mu/\sigma)/4\pi$$

Then

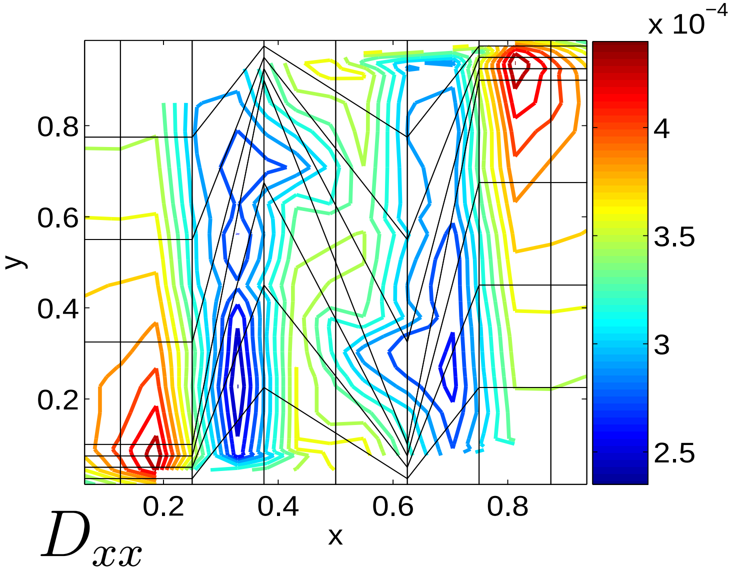
$$\mathbf{F}_{exact}(\psi_x) = -\hat{\mathbf{x}}/3\sigma = -D\hat{\mathbf{x}}$$

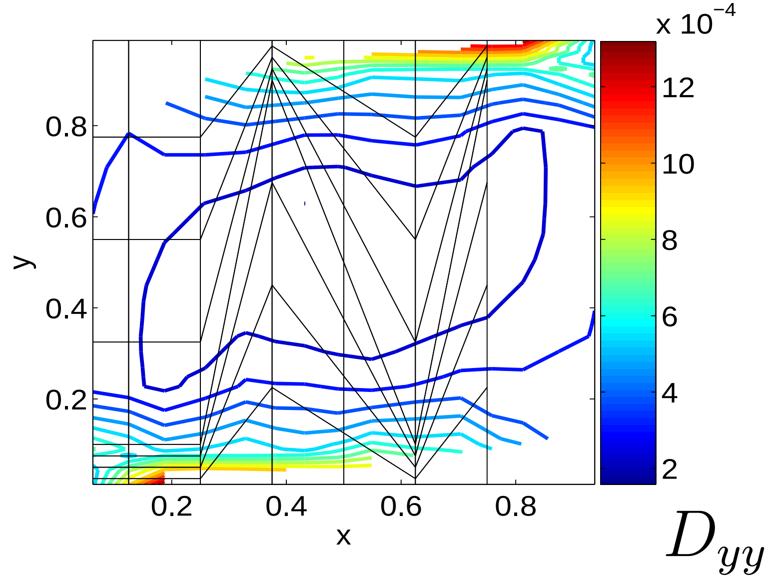
Compute approximate ψ_x and ψ_y solutions to linear problems, then calculate

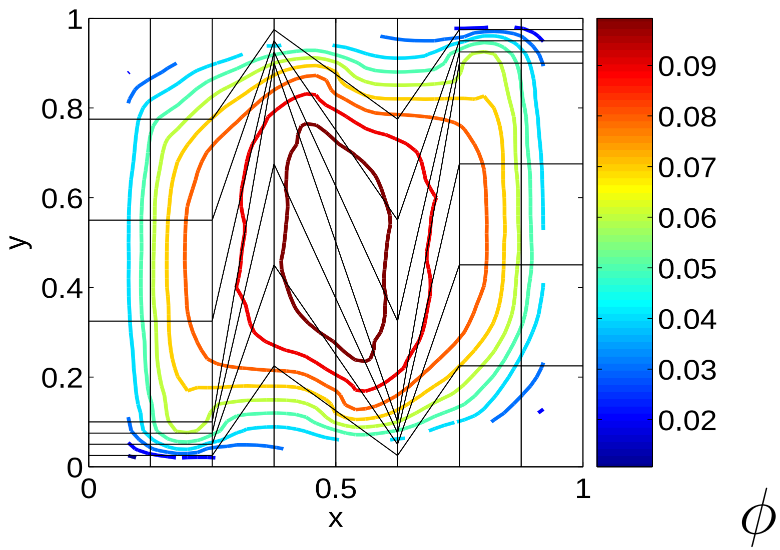
$$D_{eff} = -\begin{bmatrix} F_x(\psi_x) & F_x(\psi_y) \\ F_y(\psi_x) & F_y(\psi_y) \end{bmatrix}$$

Computational alternative to Larsen's $D(h)$ analysis.

UCB Results for $\xi = 0.1$

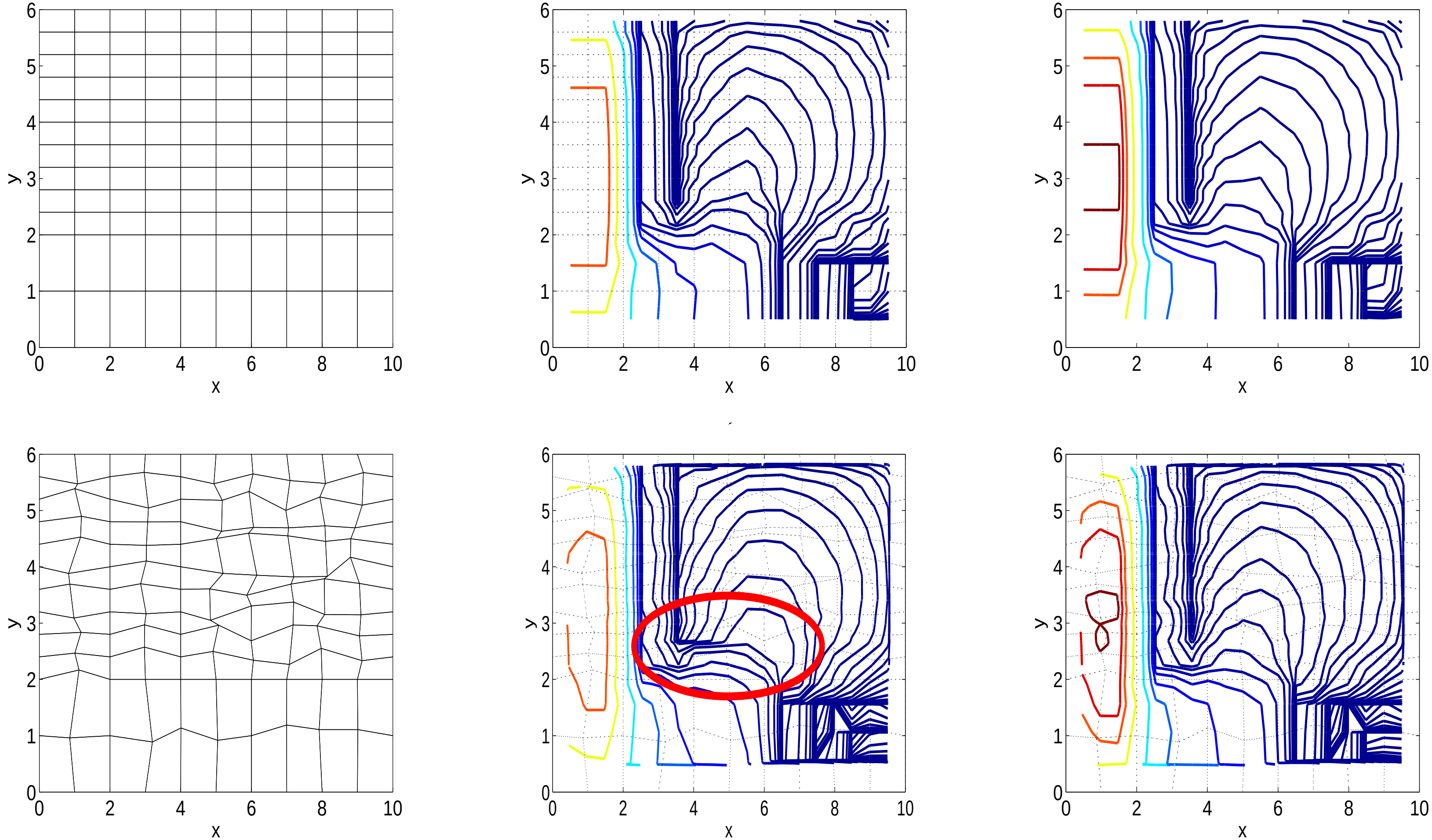






Adams “Hard” Problem

Grid UCB MMB



Interpretation

- Stand-Alone Diffusion Discretization
  - Does not have to preserve linear solutions, which may be desirable (because of loss of symmetry).
  - E.g. Morel & Shashkov's local support operators methods.
- Transport Diffusion-Limit Diffusion Discretization
  - Failure to preserve linear-solutions yields gross changes in flux.
  - This stems from the angular dependence of the linear solution.
  - D_{eff} analysis reveals this is the dominant cause of the diffusive solution degradation.
- Ideally, a transport solution will always preserves the linear solution; in the diffusion limit, this is a must!

Acknowledgments

- DOE Computational Science Graduate Fellowship
- UM Center for Advanced Computing