Global Aspects of the Book Toss

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The Euler Top

Consider an asymmetric rigid body free to rotate but, not translate with principal moments of inertia: I_1, I_2, I_3. Configuration space is $SO(3)$. Dynamics governed by conservation of ***spatial angular momentum***:

$$\dot{\boldsymbol{\pi}} = 0$$

Transform to variables fixed in the body via,

$$\boldsymbol{\pi} = \mathbf{R}\boldsymbol{\Pi}$$

where $\mathbf{R} \in SO(3)$ which represents current configuration of the body, and $\boldsymbol{\Pi}$ is the ***body angular momentum***.

Conservation of spatial angular momentum as viewed from the body yields Euler's equations,

$$\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega}$$

where $\boldsymbol{\Omega}$ is the ***body angular velocity*** which satisfies

$$\boldsymbol{\Pi} = \mathbb{I}\boldsymbol{\Omega}$$

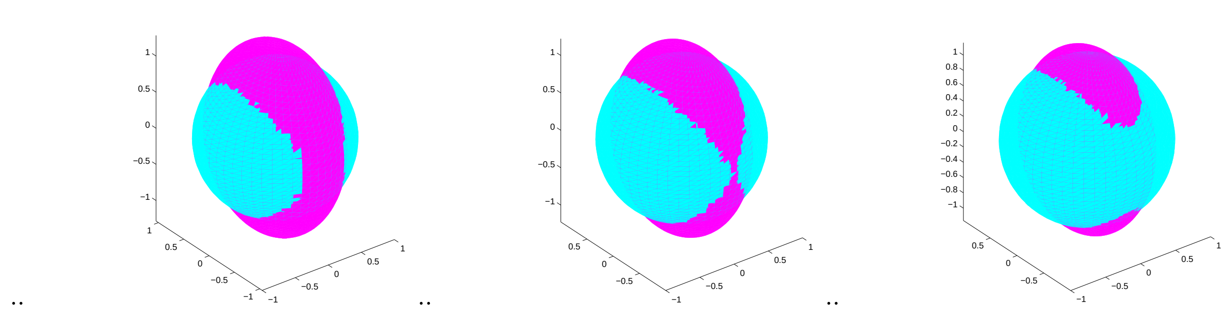
and $\mathbb{I}$ is the diagonal inertia tensor with components I_1, I_2, I_3.

Euler's equations conserve energy:

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\Omega}^T \mathbb{I} \boldsymbol{\Omega}$$

& modulus of body angular momentum:

$$\mathcal{J} = \|\boldsymbol{\Pi}\|$$



The body angular momentum vector, $\boldsymbol{\Pi}$, evolves on the intersection of the momentum sphere and energy ellipsoid as shown above for unit momentum and energy: $E = 0.225, 0.25, 0.275$.

Poinsot's Theorem

Represents motion of the free rigid body as motion of the ***inertia ellipsoid***, $\mathcal{E}$, defined as

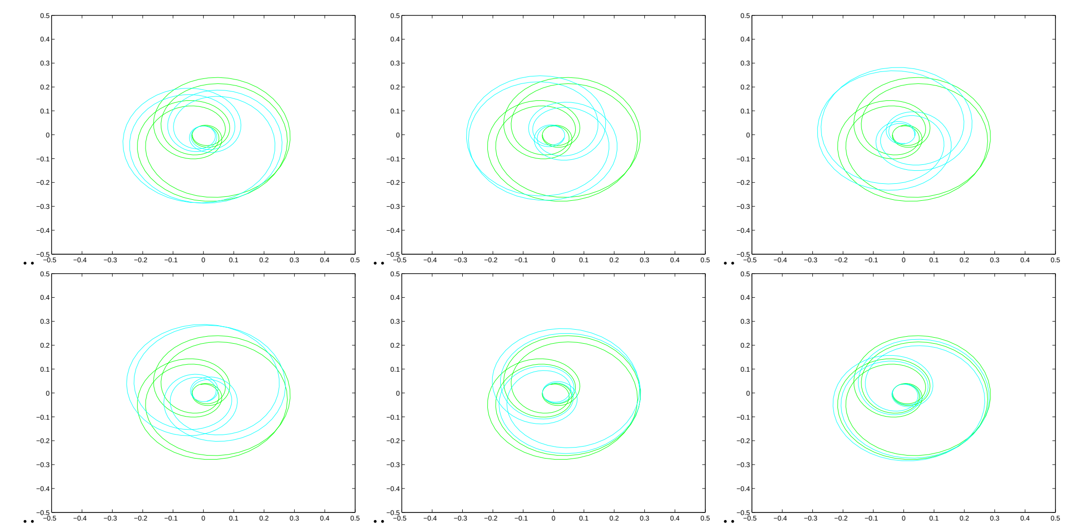
$$\mathcal{E} = \{u \in \mathbb{R}^3 : \phi(u) = u^T \mathbf{R}\mathbb{I}\mathbf{R}^T u/2 = E\}$$

with contact vector given by the ***spatial angular velocity***, $\boldsymbol{\omega} = \mathbf{R}\boldsymbol{\Omega}$, rolling on the plane whose normal is defined by the spatial angular momentum vector $\boldsymbol{\pi}$. See Marsden and Ratiu [1999] for proof.

The ***herpolhode curve*** is the trajectory of the body angular momentum vector projected on the plane with normal given by $\boldsymbol{\pi}$.

Herpolhode Curves

The body angular momentum is periodic with period T. Since, $\boldsymbol{\Pi} = \mathbb{I}\boldsymbol{\Omega}$ the body angular velocity is periodic with period T. Is the spatial angular velocity periodic with period T as well? No!



The herpolhode curve from $[0, T)$ is outlined in green and from upper left $[nT, (n+1)T)$ for $n = 1, ..., 6$ is outlined in cyan.

Since the body angular velocity is periodic with period T,

$$\boldsymbol{\Omega}(t+T) = \mathbf{R}^T(t+T)\boldsymbol{\omega}(t+T) = \mathbf{R}^T(t)\boldsymbol{\omega}(t) = \boldsymbol{\Omega}(t) \implies \boldsymbol{\omega}(t+T) = \mathbf{R}(t+T)\mathbf{R}^T(t)\boldsymbol{\omega}(t)$$

Claim: $\mathbf{R}(t+T)\mathbf{R}^T(t)$ is a rotation about the spatial angular momentum vector.

The spatial angular momentum is fixed and body angular momentum is periodic with period T, therefore,

$$\boldsymbol{\Pi}(t+T) = \mathbf{R}^T(t+T)\boldsymbol{\pi} = \mathbf{R}^T(t)\boldsymbol{\pi} = \boldsymbol{\Pi}(t) \implies \mathbf{R}(t)\mathbf{R}^T(t+T)\boldsymbol{\pi} = \boldsymbol{\pi}$$

Thus, $\mathbf{R}(t)\mathbf{R}^T(t+T)$ is a rotation about $\boldsymbol{\pi}$. What is the rotation angle?

Goodman-Robinson-Montgomery Rigid Body Phase Formula

After one period of motion, the spatial angular velocity gets rotated about the spatial angular momentum by an amount given by the ***rigid body phase formula***,

$$\Delta\theta = \frac{2ET}{M} - \Lambda \tag{1}$$

where $\mathcal{H} = E$, $\mathcal{J} = M$, T is the period of the body angular momentum, and Λ is the surface area enclosed by the intersection of the ellipsoid and sphere. See Levi [1993] for a proof based on the Gauss-Bonnet formula and Montgomery [1991] for a proof based on Stokes' theorem.

After every period of the body angular momentum, the herpolhode curve in the plane gets rotated by the amount given by the rigid body phase formula. Specifically, the herpolhode curve $\mathbf{h}(t)$ satisfies,

$$\mathbf{h}(t+T) = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \mathbf{h}(t)$$

for $t \in [0, T)$, where T is the period of the body angular momentum $\boldsymbol{\Pi}$, and $\Delta\theta$ is rigid body phase (1).

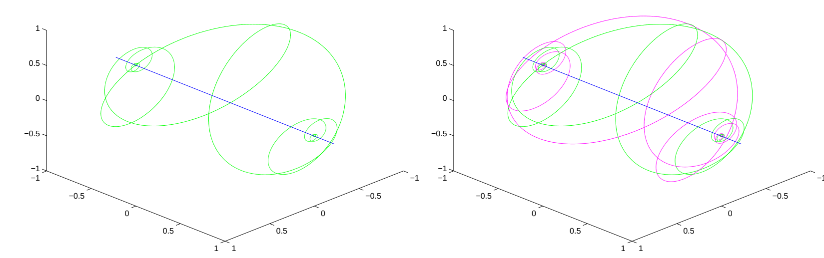
Reconstruction to Physical Variables

Is a vector fixed in the body periodic with period T? No!

After every period of the body angular momentum, an axis fixed in the rigid body is rotated about the spatial angular momentum vector by the amount given by the rigid body phase formula. Specifically, for an axis $\mathbf{n}(t)$ fixed in the rigid body,

$$\mathbf{n}(t+T) = \mathbf{R}_{\boldsymbol{\pi}}(\Delta\theta)\mathbf{n}(t)$$

for $t \in [0, T)$, where T is the period of the body angular momentum $\boldsymbol{\Pi}$, $\mathbf{R}_{\mathbf{x}}(\phi)$ is the rotation matrix with rotation axis $\mathbf{x}$ and rotation angle ϕ, and $\Delta\theta$ is the rigid body phase (1).

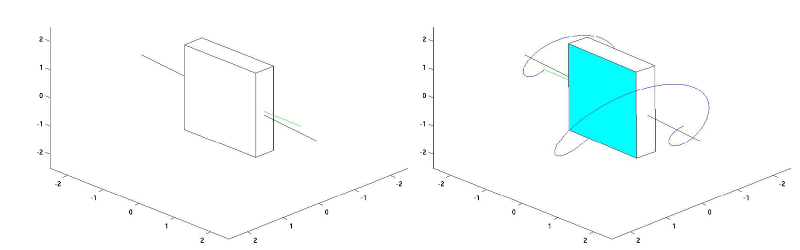


L-R: the trajectories of the intermediate principal axis of the free rigid body $\mathbf{j}(t)$ from $[0, T)$ and $[0, 2T)$.

Book Toss

When a free rigid body, e.g. a book, is tossed about its intermediate axis, the book typically undergoes a half-twist before catching it. Why does this happen?

After one cycle of the book toss or half a period of the body angular momentum $\boldsymbol{\Pi}$, the rigid body phase reconstruction theorem approximately predicts a 180-degree flip in the book as depicted below. See Ashbaugh, Chicone, and Cushman [1990] for an explanation based on special functions.



L-R: the initial and final snapshots of the book are shown for a simulation over half a period of the body angular momentum $\boldsymbol{\Pi}$. The black line represents the orientation of the spatial angular momentum vector. The green line represents the unit vector in the direction of the intermediate axis in the body.

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References

Ashbaugh, M. S., C. C. Chicone, and R. H. Cushman [1990], *Twisting Tennis Racket*. Dyn. Diff. Eqns., **3**, 67–85.

Levi, M. [1993]. *Geometric Phases in the Motion of Rigid Bodies*. Arch. Rational Mech. Anal., **122**, 213–229.

Marsden, J. E. and T. S. Ratiu [1999]. *Introduction to Mechanics and Symmetry*. Springer-Verlag, Second Edition.

Montgomery, R. [1991]. *A Berry's phase from the 18th century.*, Am. J. Phys., **59** (5), 394–398.